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Compactly supported Parseval framelets with symmetry associated to $E_d^{(2)}(\mathbb{Z})$ matrices[☆]

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ABSTRACT

Let $d \geq 1$. For any $A \in \mathbb{Z}^{d\times d}$ such that $|\det A| = 2$, we construct two families of Parseval wavelet frames with two generators. These generators have compact support, any desired number of vanishing moments, and any given degree of regularity. The first family is real valued while the second family is complex valued. To construct these families we use Daubechies low pass filters to obtain refinable functions, and adapt methods employed by Chui and He and Petukhov for dyadic dilations to this more general case. We also construct several families of Parseval wavelet frames with three generators having various symmetry properties. Our constructions are based on the same refinable functions and on techniques developed by Han and Mo and by Dong and Shen for the univariate case with dyadic dilations.

1. Introduction

Let $A \in \mathbb{Z}^{d\times d}$, $d \geq 1$, such that $|\det A| = 2$. In this article, we construct compactly supported tight wavelet frames in $L^2(\mathbb{R}^d)$ associated to A, with good properties of approximation, some symmetry, and a fixed number of generators.

A relationship between regularity and vanishing moments of a framelet and good approximation properties was shown for instance in [8]. In dimension one and with dyadic dilation, compactly supported orthonormal wavelets with any number of vanishing moments and any degree of regularity were constructed in [6]. For the multivariate case with a general dilation matrix, it is still not known if there exist compactly supported smooth wavelets with an arbitrary number of vanishing moments. As far as we know, Ref. [16] is the first paper that provides a constructive proof of the existence of tight wavelet frames associated to any general dilation matrix on $\mathbb{R}^d$ and such that their generators are compactly supported, with any given degree of regularity, any fixed number of vanishing moments, and such that the number of these generators does not depend neither on the degree of regularity nor on the number or vanishing moments. For some particular dilation matrices, such as the quincunx dilation matrix, other constructions with these properties are shown in [29,34,35].

An additional desirable property for wavelet frames is that their generators should have some symmetry. As shown in [6], in the univariate case and with the dyadic dilation, the only (up to an integer shift) compactly supported orthogonal wavelet with symmetry is the Haar wavelet. For dilation factors 3 and 4, symmetric/antisymmetric orthogonal wavelets

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are constructed in [5] and [14], respectively. When the dilation factor is larger than 2, a method for constructing symmetric orthogonal scaling functions is suggested in [2]. In the same context, for a low pass filter associated to an orthogonal scaling function, an algorithm to obtain unitary matrices with symmetry is given in [31]. The case of orthogonal symmetric refinable masks with complex coefficients, linear–phase moments, order of sum rule and step–by–step algorithm for construction high pass filters was investigated in [18–20].

In [33], it is shown that given any B–spline function of order m it is possible to construct a symmetric tight wavelet frame with m generators. For a given symmetric refinable function, it is shown in [3] that it is possible to obtain a symmetric tight wavelet frame with three generators, and in [31] it is observed that in some cases the number of generators can be reduced to 2. The authors of [1] developed a method for constructing compactly supported (anti)symmetric wavelet frames with two generators and a few vanishing moments. Other constructions with the additional property of having many vanishing moments appeared in [23] (see [22]). Symmetric compactly supported Parseval wavelet frames with three generators were constructed using pseudo-splines and the unitary extension principle in [10]. For an arbitrary dilation factor larger than 2, algorithms for constructing (anti)symmetric tight wavelet frames generated by a given refinable function were presented in [32]. Starting from any two symmetric compactly supported refinable functions in $L^2(\mathbb{R})$ with dilation factor M, it is shown in [9] that it is always possible to construct $2M$ wavelet functions with compact support that generate a pair of (anti)symmetric dual wavelet frames in $L^2(\mathbb{R})$.

In the multivariate case and with a general matrix dilation, symmetric refinable scaling functions were studied in [15,17]. In [11], a general and simple method is provided. The number of generators for the symmetric tight wavelet frames is at most twice the number of generators for the original tight wavelet frame, and the number of vanishing moments is preserved. In [25], for arbitrary marix dilations whose determinant is either odd or equal to ± 2, all real interpolatory masks which are symmetric with respect to the origin and generate symmetric/antisymmetric compactly supported biorthonormal bases or dual wavelet systems, are described. Systematic methods for completion of unitary matrices with symmetric trigonometric polynomial when the first row is given are developed in [18,21], whereas [26] is devoted to the study and construction of point symmetric refinable masks and point symmetric/antisymmetric frame-like wavelet system with desired approximation properties.

Many of the results cited above are discussed in the book [24] by Krivoshein et al.

We will focus on dilation matrices $A \in \mathbb{R}^{d\times d}$, $d \geq 1$, with integer entries and $|\det A| = 2$. One of the main difficulties for the construction of wavelet frames in the multivariate case is that there is no analog of the lemma of Riesz that affirms that given a real valued trigonometric polynomial there is a trigonometric polynomial such that the square of its absolute value is the given polynomial. Thus, perhaps our main advance here is the observation that from the structure of the quotient group $\mathbb{Z}^d/A\mathbb{Z}^d$, techniques developed by Han and Mo and by Dong and Shen for the univariate case with dyadic dilation can be used to construct real and complex valued smooth compactly supported Parseval wavelet frames associated to A with many vanishing moments and some symmetry.

We consider $A \in \mathbb{R}^{d\times d}$, $d \geq 1$, a dilation matrix with integer entries and $|\det A| = 2$. In this paper, basic definitions and notation are enclosed in Section 2. Section 3 is devoted to exhibit a family of real valued Parseval wavelet frames associated to A with two generators where the generator functions have compact support, any desired number of vanishing moments and any given degree of regularity. This family of Parseval framelets is constructed in Theorem 2 of [43], but our innovation here is that we show that they can have high degree of regularity. In Section 4, we construct two families of real valued Parseval wavelets associated to A with the additional property that the generators have some symmetry. In this case, we need three generators instead of two. In Sections 5 and 6 we use the same techniques as in Sections 3 and 4 to construct families of compactly supported complex-valued Parseval wavelet frames associated to A with analogous properties of regularity, number of vanishing moments and symmetry.

Although our constructions provide explicit formulas for generators having any given number of vanishing moments (and therefore any given degree of regularity), these are theoretical results of difficult practical implementation. We intend to study this problem further in a forthcoming paper.

The articles [27,28] by Krivoshian and [36,37] by Skopina are closely related to our paper.

2. Notation and basic definitions

We now introduce our notation and basic definitions. The sets of strictly positive integers, integers, rational numbers, real numbers and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We will write $\mathbf{t} = (t_1, \ldots, t_d)^T \in \mathbb{R}^d$ and $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$. If $r \in \mathbb{C}$ then $r\mathbf{t} = (rt_1, \ldots, rt_d)^T$ and $r\mathbf{x} = (rx_1, \ldots, rx_d)^T$. We will assume that $n, m \in \mathbb{N}$.

Given a matrix A, its transpose will be denoted by A^T, and the conjugate of its transpose by A^*. If we write $\mathbf{I}_d$ we mean the $d \times d$ identity matrix.

We say that $A \in \mathbb{R}^{d\times d}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$ if all its eigenvalues have modulus greater than 1 and $A\mathbb{Z}^d \subset \mathbb{Z}^d$. The set of all $d \times d$ dilation matrices preserving the lattice $\mathbb{Z}^d$ will be denoted by $\mathbf{E}_d(\mathbb{Z})$. Note that if $A \in \mathbf{E}_d(\mathbb{Z})$ then $|\det A|$ is an integer greater than 1. Here we will focus on the subset of matrices $A \in \mathbf{E}_d(\mathbb{Z})$ such that $|\det A| = 2$. This set will be denoted by $\mathbf{E}_d^{(2)}(\mathbb{Z})$. If A is in $\mathbf{E}_d(\mathbb{Z})$, the quotient groups $\mathbb{Z}^d/A\mathbb{Z}^d$ and $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ are well defined. From e.g. [13, Lemma 2] we know that $\mathbb{Z}^d/A\mathbb{Z}^d$ has exactly $|\det A|$ cosets, which readily implies that also $A^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ has exactly $|\det A|$ cosets.