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Numerical solution of degenerate stochastic Kawarada equations via a semi-discretized approach

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ABSTRACT

The numerical solution of a highly nonlinear two-dimensional degenerate stochastic Kawarada equation is investigated. A semi-discretized approximation in space is comprised on arbitrary nonuniform grids. Exponential splitting strategies are then applied to advance solutions of the semi-discretized scheme over adaptive grids in time. It is shown that key quenching solution features including the positivity and monotonicity are well preserved under modest restrictions. The numerical stability of the underlying splitting method is also maintained without any additional restriction. Computational experiments are provided to not only illustrate our results, but also provide further insights into the global nonlinear convergence of the numerical solution.

1. Introduction

Kawarada partial differential equations arise in the modeling of highly sophisticated, yet important, natural phenomena where singularities may develop as the solution evolves in time. Such singularities often result from the energy of a system concentrating and approaching its activation criterion [1,2].

Consider a typical solid-fuel ignition process. If the combustion chamber with fuel and air are appropriately mixed, the temperature in the chamber may increase monotonically until a certain critical value is reached. However, rates of such temperature changes can occur in a nonlinear manner throughout the media. This nonuniform distribution of heat may result in high temperatures being extremely localized within the combustion chamber and lead to an ignition once the peak temperature reaches a certain threshold [1]. This phenomenon is carefully characterized by a Kawarada model in which temporal derivatives of the solution may grow at an explosive rate, while the solution itself remains bounded [3–6]. This strong nonlinear singularity, which is referred to as a *quenching singularity*, has been the backbone of the modeling equations. Applications of Kawarada equations can be found in fuel combustion simulations, enhanced thermionic emission optimization processes, electric current transients in polarized ionic chambers, and enzyme kinematics.

Let $\mathcal{D} = (-a, a) \times (-b, b) \subset \mathbb{R}^2$ be an idealized combustion chamber, $\partial\mathcal{D} = \bar{\mathcal{D}} \setminus \mathcal{D}$ be its boundary, and $\Omega = \mathcal{D} \times (t_0, T_q)$, $\mathcal{S} = \partial\mathcal{D} \times (t_0, T_q)$, where $a,\ b > 0, 0 \le t_0 < T_q < \infty$ with T_q being the terminal time. Further, let $c > 0$ be the ignition temperature. If $u(x, y, t)$ denotes the temperature distribution within the chamber, we have the following 2D model generalized from the original Kawarada configuration [4]:

$$\mathscr{L}u = f(\varepsilon, u), \quad (x, y, t) \in \Omega, \tag{1.1}$$

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$$u(x, y, t) = 0, \quad (x, y, t) \in \mathcal{S}, \tag{1.2}$$

$$u(x, y, t_0) = u_0(x, y) \ll c, \quad (x, y) \in \mathcal{D}, \tag{1.3}$$

where

$$\mathscr{L} := \sigma(x, y)\partial_t - \Delta,$$

and Δ is the standard 2D Laplacian. The degeneracy function $\sigma(x, y) \geq 0$ for $x \in \bar{\mathcal{D}}$ with equality occurring only for $(x, y) \in \partial\mathcal{D}_0 \subseteq \partial\mathcal{D}$. The nonlinear reaction term, $f(\varepsilon, u)$ with a bounded stochastic influence $\varepsilon(x, y)$, satisfies the following properties

$$f(\varepsilon, 0) = f_0(\varepsilon) > 0, \quad f_u(\varepsilon, u) > 0, \quad \lim_{u \to c^-} f(\varepsilon, u) = \infty, \quad \int_0^c f(\varepsilon, u)\, du = \infty$$

for $u \in [0, c)$. The stochastic influence is characterized by

$$0 < f_{\min}^{\varepsilon} \leq \min_{\varepsilon}\{f(\varepsilon, u)\} \quad \text{and} \quad \max_{\varepsilon}\{f(\varepsilon, u)\} \leq f_{\max}^{\varepsilon} \ll \infty$$

for each $u \in [0, c)$. The solution u of (1.1)–(1.3) is said to *quench* if there exists $T_q < \infty$ such that

$$\sup\{u_t(x, y, t) \,:\, (x, y) \in \mathcal{D}\} \to \infty \text{ as } t \to T_q^-. \tag{1.4}$$

T_q is then referred to as the *quenching time* [3,7,8]. It has been shown that a necessary condition for quenching to occur with the above conditions placed on the reaction term is

$$\max\{u(x, y, t) \,:\, (x, y) \in \bar{\mathcal{D}}\} \to c^- \text{ as } t \to T_q^-. \tag{1.5}$$

We may note that quenching is a mathematical interpretation of the fuel ignition/combustion process in physics. Since $T_q < \infty$ only when certain spatial references, such as the size and shape of $\mathcal{D}$, reach their critical limits, a domain Ω_* is called a *critical domain* of (1.1)–(1.3) if the solution of (1.1)–(1.3) exists globally as $\Omega \subseteq \Omega_*$ and (1.5) occurs as $\Omega_* \subseteq \Omega$. We note that such critical domains are not unique in multi-dimensional circumstances, since both the *size* and *shape* of Ω affect combustion [9–12].

Highly effective and efficient algorithms for solving Kawarada equations such as (1.1)–(1.3) have been difficult to obtain due to the highly nonlinear mechanism. There has been much effort to develop algorithms that may accurately predict the ignition location and time, solution profiles, and critical domain characteristics, while reasonably conserving the solution positivity and monotonicity [12–17]. While explorations have been carried out on both uniform and nonuniform grids, as well as via adaptations, rigorous analysis of different schemes involving nonuniform grids has been incomplete. To improve computational efficiency, splitting strategies are introduced and incorporated with adaptive mesh structures [11,13,18]. The success of such combinations has also become a key motivation for the study to be presented in this paper.

Rigorous mathematical analysis is implemented for a highly vibrant semi-discretization oriented method for solving (1.1)–(1.3) in our study. In the next section, we verify carefully the order of accuracy of the scheme proposed. In Sections 3 and 4, we prove that the solution of the semi-discretized system, as well as the corresponding scheme, preserve the expected solution positivity and monotonicity under reasonable restrictions. Section 5 is devoted to standard and reinforced stability analysis of the numerical method. Experimental results to validate our analysis and explore global nonlinear convergence of the stochastic singular Kawarada solutions are presented in Section 6. Finally, Section 7 is tailored for concluding our remarks, concerns, and forthcoming endeavors.

2. Derivation of scheme

Without loss of generality, we set $c = 1$. Utilizing transformations $\tilde{x} = x/a$, $\tilde{y} = y/b$, and reusing the original variables for notation simplicity, we may reformulate (1.1)–(1.3) as

$$u_t = \frac{1}{a^2\sigma}u_{xx} + \frac{1}{b^2\sigma}u_{yy} + g(\varepsilon, u), \quad (x, y, t) \in \Omega, \tag{2.1}$$

$$u(x, y, t) = 0, \quad (x, y, t) \in \mathcal{S}, \tag{2.2}$$

$$u(x, y, t_0) = u_0(x, y) \ll 1, \quad (x, y) \in \mathcal{D}, \tag{2.3}$$

where $\mathcal{D} = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ and $g(\varepsilon, u) = f(\varepsilon, u)/\sigma(x, y)$. For fixed N_x, $N_y \gg 1$ we define

$$\mathcal{D}_h = \left\{(x_i, y_j) \,:\, 0 \leq i \leq N_x + 1;\ 0 \leq j \leq N_y + 1\right\},$$
$$\mathcal{D}_h^{\circ} = \left\{(x_i, y_j) \,:\, 0 < i < N_x + 1;\ 0 < j < N_y + 1\right\},$$