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Stepwise irregular graphs

Ivan Gutman

Faculty of Science, University of Kragujevac, Kragujevac, Serbia

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ABSTRACT

A graph is stepwise irregular (SI) if the degrees of any two of its adjacent vertices differ by exactly one. Among graphs with non-zero edge imbalance, SI graphs are least irregular. Some basic properties of SI graphs are established.

1. Introduction

Graphs in which all vertices have equal degrees are said to be regular. Regular graphs played and outstanding role in the history of graph theory [9], and are equally important at the present time. Graphs that are not regular may be referred to as *irregular*. It seems that the irregularity of graphs was first examined in the 1980s by Erdős et al. [5,10,11]. This soon made it necessary to design a quantitative measure of irregularity. The first such irregularity index was put forward by Bell [8], although a spectrum-based irregularity measure can be recognized already in the seminal paper by Collatz and Sinogowitz [13]. Nowadays, the most popular and most frequently used irregularity index is that of Albertson [6], defined as

$$Alb(G) = \sum_{uv \in \mathbf{E}(G)} |d(u) - d(v)| \,, \tag{1}$$

where $d(u)$ is the degree of the vertex u, and where $|d(u) - d(v)|$ is the *imbalance* of the edge uv connecting the vertices u and v. The summation in Eq. (1) goes over all pairs of adjacent vertices of the underlying graph G.

The Albertson index was applied to a variety of classes of graphs and found numerous applications (see the recent works [1,2,4,18,19,21,22,26] and the references quoted therein).

It should be mentioned that in the mathematical literature, several other irregularity indices have also been investigated [3,12,16,23].

In the present paper, we focus our attention to a special class of graphs, in which the relation $|d(u) - d(v)| = 1$ holds for all pairs of adjacent vertices. Such graphs may be viewed as having minimum irregularity among graphs with non-zero edge imbalance. We propose to call such graphs *stepwise irregular*, or shorter, SI.

In what follows, in order to avoid unnecessary complications, we assume that the graphs considered are simple and connected. Let G be such a graph, with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$. Let $|\mathbf{V}(G)| = n$ and $|\mathbf{E}(G)| = m$. If u, v are adjacent vertices of G, then the edge connecting them is denoted by uv.

The degree (= number of first neighbors) of a vertex $u \in \mathbf{V}(G)$ will be denoted by $d(u)$. In addition, $\Delta = \max_{u \in \mathbf{V}(G)} d(u)$.

The following simple result is found already in Albertson's seminal paper [6].

Proposition 1 [6]**.** *The Albertson index, Eq. (1), of any graph G is an even integer.*

E-mail address: gutman@kg.ac.rs

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Proof. For any edge $uv \in \mathbf{E}(G)$, the parity of the edge imbalance $|d(u)-d(v)|$ is same as the parity of $\big[d(u)-d(v)\big]^2$. Therefore, the parity of the Albertson index is same as the parity of

$$\sum_{uv\in\mathbf{E}(G)} \big[d(u)-d(v)\big]^2 = \sum_{uv\in\mathbf{E}(G)} \big[d(u)^2+d(v)^2\big] - 2\sum_{uv\in\mathbf{E}(G)} d(u)\,d(v)$$
$$= \sum_{u\in\mathbf{V}(G)} d(u)^3 - 2\sum_{uv\in\mathbf{E}(G)} d(u)\,d(v)\,.$$

Since the sum of vertex degrees is equal to $2m$, the term $\sum_u d(u)^3$ must be an even integer. The term $2\sum_{uv} d(u)\,d(v)$ is evidently even. Therefore $\sum_{uv} \big[d(u)-d(v)\big]^2$ is even and therefore $Alb(G)$ is even. □

The distance $d(u,v|G)$ between two vertices u and v of the graph G is the length of (= number of edges in) a shortest path connecting u and v. The classical Wiener index of the graph G is then defined as [24,25]

$$W(G) = \sum_{\{u,v\}\subseteq\mathbf{V}(G)} d(u,v|G)\,. \tag{2}$$

Two other degree-and-distance-based graph invariants have been much studied in the recent literature. These are the degree distance [14,15,17]

$$DD(G) = \sum_{\{u,v\}\subseteq\mathbf{V}(G)} \big[d(u)+d(v)\big] d(u,v|G) \tag{3}$$

and

$$ZZ(G) = \sum_{\{u,v\}\subseteq\mathbf{V}(G)} \big[d(u)\cdot d(v)\big] d(u,v|G) \tag{4}$$

which sometimes is referred to as the Gutman index [7,14,17,20].

2. Simple results

Directly from the definition of an SI graph, it follows that $Alb(G)=m$. Then by Proposition 1 we get:

Lemma 2. *The number of edges of a stepwise irregular graph is even.*

Lemma 3. *Stepwise irregular graphs are bipartite.*

Proof. It is sufficient to show that an SI graph cannot possess a cycle of odd size.

Let C be a cycle of an SI graph, and u a vertex belonging to C. Start at vertex u and go along the cycle C. In each step, the degrees of the vertices increase or decrease by one. After closing the cycle, the vertex degree must again be equal to $d(u)$, which requires an equal number of additions and subtractions, i.e., the size of the cycle C must be even. □

Evidently, if G is an SI graph, then it is bipartioned into the set of vertices of even degree and those of odd degree. This immediately implies:

Lemma 4. *In a stepwise irregular graph, the distance between two vertices whose degrees are of equal parity is even, whereas the distance between two vertices whose degrees are of opposite parity is odd.*

Lemma 4 has the following direct consequence:

Theorem 5. *If G is a stepwise irregular graph, then the graph invariants $W(G)$, $DD(G)$ and $ZZ(G)$, Eqs. (2)–(4), are even integers.*

Proof. Let the number of even-degree and odd-degree vertices of G be n_e and n_o, respectively, $n_e+n_o=n$. Since the sum of vertex degrees is an even number ($=2m$), n_o must be even.

Consider the Wiener index, Eq. (2). The sum of distances can be divided into three parts: (1) distances between pairs of even-degree vertices, (2) distances between pairs of odd-degree vertices and (3) distances between vertices of degrees of different parity. By Lemma 4, the first and second parts must be even. The summands in the third part are odd-valued, but their number is $n_e\cdot n_o$, an even number. Therefore also the third part is an even number.

The proofs of Theorem 5 for $DD(G)$ and $ZZ(G)$ are analogous. □

3. The order of stepwise irregular graphs

We first prove an auxiliary result.

Lemma 6. *Let G_0 be a stepwise irregular graph whose vertex u is of degree 1, cf. Fig. 1. Construct the graph G_1 by attaching two two-vertex paths to the vertex u, cf. Fig. 1. Then also G_1 is a stepwise irregular graph.*