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Exact solutions via equivalence transformations of variable-coefficient fifth-order KdV equations

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ABSTRACT

In this paper, a family of variable-coefficient fifth-order KdV equations has been considered. By using an infinitesimal method based on the determination of the equivalence group, differential invariants and invariant equations are obtained. Invariants provide an alternative way to find equations from the family which may be equivalent to a specific subclass of the same family and the invertible transformation which maps both equivalent equations. Here, differential invariants are applied to obtain exact solutions.

1. Introduction

In this paper, we are interested to look for exact solutions via equivalence transformations of fifth-order KdV equations with time-dependent coefficients and linear damping term of the form

$$u_t + A(t)u_{xxxxx} + B(t)u_{xxx} + C(t)uu_{xxx} + E(t)uu_x + F(t)u_xu_{xx} + Q(t)u = 0, \tag{1}$$

where $A(t)\neq 0$, $B(t)$, $C(t)\neq 0$, $E(t)$, $F(t)$ and $Q(t)$ are arbitrary smooth functions of t.

We recall that an equivalence transformation of class (1) is a nondegenerate transformation, which acts on independent and dependent variables given by

$$t = f(\tilde{t}, \tilde{x}, \tilde{u}), \quad x = g(\tilde{t}, \tilde{x}, \tilde{u}), \quad u = h(\tilde{t}, \tilde{x}, \tilde{u}), \tag{2}$$

in a manner that it transforms Eq. (1) in

$$\tilde{u}_{\tilde{t}} + \tilde{A}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \tilde{B}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{C}(\tilde{t})\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{E}(\tilde{t})\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{F}(\tilde{t})\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{Q}(\tilde{t})\tilde{u} = 0,$$

i.e., it maps Eq. (1) to another equation of the same class, where the transformed functions can be different from the original ones. The equivalence transformations have important applications. For instance, they can be used to determine equivalent formulations of a class of partial differential equations (PDEs) that could simplify the analysis of the class, especially if one expect to perform a classification problem. With respect to the concept of equivalence transformations, some results have been communicated in the recent literature [6–8,10–12,16,29,33].

Nonlinear evolution equations (NLEEs) have been studied in a great number of works from the similarity reductions (see e.g. [1,2,5,9,18,20,21,25,27,34]) which lead the NLEEs to ordinary differential equations (ODEs). However, it is not always evident how to integrate these ODEs. A different way is to consider another equation with a known solution, which is

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related with the equation under consideration through an invertible transformation. For a class of PDEs, this equivalence problem may be solved by using the differential invariants of its group of equivalence transformations.

A differential invariant of a class of PDEs is a real valued function which is invariant under the equivalence group. The applications of differential invariants of the Lie groups of continuous transformations can be found in an extensive variety of problems arising in mathematical modeling, differential equations, differential geometry and nonlinear science. There exist different methods to approach the equivalence problem. Among them, the Lie infinitesimal method is one of the most frequently used. Based on Lie's preliminary results [17], Ovsyannikov [24] developed this approach for infinite Lie groups. Furthermore, this method was subsequently used for obtaining equivalence transformations. Afterwards, in [23], Olver presented the equivariant method of moving frames which is used to generate differential invariants for finite dimensional Lie group actions. A broad theory on differential invariants of Lie groups along with algorithms of construction of differential invariants is given in [22,24].

Equivalence transformations play an important role in the theory of differential invariants. Indeed, the derivation of equivalence transformations of a class of PDEs represents the initial step towards the determination of differential invariants. In [13,14], Ibragimov proposed a method for calculation of differential invariants of classes of linear and nonlinear equations which have an infinite dimensional equivalence group. Following Ibragimov's method several authors have constructed differential invariants, see e.g. [3,4,19,26,31]. Furthermore, the knowledge of differential invariants can be used to find special equivalence transformations that linearize some classes of equations (see e.g. [15,28,30]).

This work is organized as follows. In the next section, we show the equivalence transformations of class (1) and of its subclass

$$u_t + A(t)u_{xxxxx} + B(t)u_{xxx} + C(t)uu_{xxx} + E(t)uu_x + F(t)u_xu_{xx} = 0. \tag{3}$$

In Section 3, following Ibragimov's method, we get differential invariants of zero and first order for class (3). As an application of the invariants obtained, in Section 4, we determine exact solutions for special subclasses of family (1).

2. Equivalence transformations

An equivalence transformation of class (1) is a nondegenerate point transformation from (t, x, u) to $(\tilde{t}, \tilde{x}, \tilde{u})$ that preserves the differential structure of the equation, that is, it maps every equation of class (1) into another equation of the same class but with different functions, $\tilde{A}(\tilde{t}),\ \tilde{B}(\tilde{t}),\ \tilde{C}(\tilde{t}),\ \tilde{E}(\tilde{t}),\ \tilde{F}(\tilde{t})$ and $\tilde{Q}(\tilde{t})$. The set of equivalence transformations forms a group. Generators of the group of equivalence transformations of Eq. (1) take the form

$$Y = \tau\partial_t + \xi\partial_x + \eta\partial_u + \omega_1\partial_A + \omega_2\partial_B + \omega_3\partial_C + \omega_4\partial_E + \omega_5\partial_F + \omega_6\partial_Q. \tag{4}$$

In [7] it was proved that class (1) admits an infinite dimensional equivalence group $\mathcal{G}$ whose generators are given by

$$Y_1 = x\partial_x + 5A\partial_A + 3B\partial_B + 3C\partial_C + E\partial_E + 3F\partial_F, \tag{5}$$

$$Y_2 = \partial_x, \tag{6}$$

$$Y_\alpha = \alpha\partial_t - \alpha_tA\partial_A - \alpha_tB\partial_B - \alpha_tC\partial_C - \alpha_tE\partial_E - \alpha_tF\partial_F - \alpha_tQ\partial_Q, \tag{7}$$

$$Y_\beta = \beta u\partial_u - \beta C\partial_C - \beta E\partial_E - \beta F\partial_F - \beta_t\partial_Q, \tag{8}$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are arbitrary smooth functions of t. From generators (5)–(8) the finite form of the equivalence transformations was obtained

$$\tilde{t} = \lambda(t), \quad \tilde{x} = (x + k_2)e^{k_1}, \quad \tilde{u} = e^{\frac{k_1}{2} + k_3\mu(t)}u, \tag{9}$$

where k_1, k_2 and k_3 are arbitrary constants, $\lambda = \lambda(t)$ and $\mu = \mu(t)$ are arbitrary smooth functions with $\lambda_t \neq 0$.

We observe that the transformation

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = e^{\int Q(t)dt}u, \tag{10}$$

maps equations of the subclass

$$\tilde{u}_{\tilde{t}} + \tilde{A}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \tilde{B}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{C}(\tilde{t})\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{E}(\tilde{t})\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{F}(\tilde{t})\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{x}} = 0, \tag{11}$$

in an equation of class (1).

The subclass (3) admits an infinite dimensional equivalence group $\mathcal{G}'$ whose generators are given by

$$Y_1 = x\partial_x + 5A\partial_A + 3B\partial_B + 3C\partial_C + E\partial_E + 3F\partial_F, \tag{12}$$

$$Y_2 = \partial_x, \tag{13}$$

$$Y_3 = u\partial_u - C\partial_C - E\partial_E - F\partial_F, \tag{14}$$

$$Y_\alpha = \alpha\partial_t - \alpha_tA\partial_A - \alpha_tB\partial_B - \alpha_tC\partial_C - \alpha_tE\partial_E - \alpha_tF\partial_F, \tag{15}$$