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Odd graph and its applications to the strong edge coloring

Tao Wang*, Xiaodan Zhao

Institute of Applied Mathematics Henan University, Kaifeng 475004, PR China

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ABSTRACT

A strong edge coloring of a graph is a proper edge coloring in which every color class is an induced matching. The strong chromatic index $\chi_s'(G)$ of a graph G is the minimum number of colors in a strong edge coloring of G. Let $\Delta \geq 4$ be an integer. In this note, we study the odd graphs and show the existence of some special walks. By using these results and Chang's et al. (2014) ideas, we show that every planar graph with maximum degree at most Δ and girth at least $10\Delta - 4$ has a strong edge coloring with $2\Delta - 1$ colors. In addition, we prove that if G is a graph with girth at least $2\Delta - 1$ and $\mathrm{mad}(G) < 2 + \frac{1}{3\Delta-2}$, where Δ is the maximum degree and $\Delta \geq 4$, then $\chi_s'(G) \leq 2\Delta - 1$; if G is a subcubic graph with girth at least 8 and $\mathrm{mad}(G) < 2 + \frac{2}{23}$, then $\chi_s'(G) \leq 5$.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. A *strong edge coloring* of a graph is a proper edge coloring in which every color class is an induced matching. The *strong chromatic index* $\chi_s'(G)$ of a graph G is the minimum number of colors in a strong edge coloring of G. Fouquet and Jolivet [12,13] introduced the notion of strong edge coloring for the radio networks and their frequencies assignment problem.

The greedy algorithm provides an upper bound $2\Delta(\Delta-1)+1$ on the strong chromatic index, where Δ is the maximum degree of the graph. In 1985, Erdős and Nešetřil constructed graphs with strong chromatic index $\frac{5}{4}\Delta^2$ when Δ is even, $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ when Δ is odd. Inspired by their construction, they proposed the following strong edge coloring conjecture during a seminar in Prague.

Conjecture 1 (Erdős and Nešetřil, see [10]). *If G is a graph with maximum degree Δ, then*

$$\chi_s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even,} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

For $\Delta = 3$, the conjecture was confirmed independently by Horák et al. [14] and Andersen [1]. For $\Delta = 4$, Cranston [7] showed that 22 suffices, and this was further improved to 21 by Huang et al. [15]. In 1997, Molloy and Reed [17] gave the first general bound by using probabilistic techniques, showing that if Δ is large enough, then every graph with maximum degree Δ has a strong edge coloring with $1.998\Delta^2$ colors. Recently, the upper bound was improved to $1.93\Delta^2$ by Bruhn and Joos [4]. Very recently, this was further improved to $1.835\Delta^2$ by Bonamy et al. [2].

For planar graphs, Faudree et al. [11] established the following upper bound using the Four Color Theorem and Vizing's Theorem.

* Corresponding author.
E-mail address: wangtao@henu.edu.cn (T. Wang).

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Theorem 1.1 (Faudree et al. [11]). *Every planar graph with maximum degree* Δ *has a strong edge coloring with* $4\Delta+4$ *colors.*

Kostochka et al. [16] showed that every subcubic planar multigraph without loops (multiple edges are allowed) has a strong edge coloring with nine colors. Borodin and Ivanova [3] showed that if G is a planar graph with girth at least $40\lfloor\frac{\Delta}{2}\rfloor+1$, then $\chi_s'(G)\le 2\Delta-1$. Chang et al. [6] improved the lower bound on the girth as the following.

Theorem 1.2 (Chang et al. [6]). *Let* $\Delta\ge 4$ *be an integer. If* G *is a planar graph with maximum degree at most* Δ *and girth at least* $10\Delta+46$, *then* $\chi_s'(G)\le 2\Delta-1$.

In this note, we improve the lower bound on the girth to $10\Delta-4$ when $\Delta\ge 4$, by following and improving the method in [6]. In addition, we give two results with the condition on the maximum average degree. The maximum average degree $\mathrm{mad}(G)$ of a graph G is defined as

$$\mathrm{mad}(G)=\max_{\emptyset\ne H\subseteq G}\left\{\frac{2|E(H)|}{|V(H)|}\right\}.$$

2. Odd graphs

The odd graph O_n has one vertex for each of the $(n-1)$-element subsets of a $(2n-1)$-element set $\{1,2,\ldots,2n-1\}$. Two vertices are adjacent if and only if the corresponding subsets are disjoint. O_n is a Kneser graph $KG(2n-1,n-1)$. We use v to denote a vertex in the odd graph, and use $[v]$ to denote the corresponding subset. By the definition, every edge uv can be labeled with a unique element which is not contained in $[u]$ or $[v]$. By the particular structure of odd graphs, the labels on the edges induce a strong edge coloring of the odd graph. We will use this property in the proof of Theorem 3.2. A *walk* is a sequence of edges such that every two consecutive edges are adjacent. A *special walk* is a walk whose every two consecutive edges are distinct. A *k-special walk* is a special walk of length k. A *k-path* is a path with k edges.

2.1. Structural results

Lemma 1. *Let* O_n *be the odd graph. Then the following items hold:*

(a) *If the distance between* u *and* v *is two, then* $[u]$ *and* $[v]$ *have exactly one different element. Thus, for two vertices* u *and* v, *if* $[u]$ *and* $[v]$ *have exactly* k *different elements, then they can be reached from each other in* $2k$ *steps and this path is a shortest even path.*

(b) *If* $w_1w_2w_3$ *is a path with* w_1w_2 *and* w_2w_3 *being labeled with* x *and* y *respectively, then* $[w_3]$ *is obtained from* $[w_1]$ *by replacing* y *with* x.

(c) *Every 3-path* $v_1v_2v_3v_4$ *is contained in a 6-cycle.*

Proof. The first two items are trivial, so we do not include their proofs. Note that the sets corresponding to v_1 and v_3 differ only on one element. So we may assume that $[v_1]=X\cup\{x_1\}$ and $[v_3]=X\cup\{x_2\}$, where X is an $(n-2)$-subset of $\{1,2,\ldots,2n-1\}$. Thus, we can obtain $[v_2]=\{1,2,\ldots,2n-1\}\setminus(X\cup\{x_1,x_2\})$. Without loss of generality, we may assume that $[v_4]=\{1,2,\ldots,2n-1\}\setminus(X\cup\{x_2,x_3\})$ and $x_1\ne x_3$. It is easy to check that the path $v_1v_2v_3v_4$ is contained in a 6-cycle $v_1v_2v_3v_4v_5v_6$, where $[v_5]=X\cup\{x_3\}$ and $[v_6]=\{1,2,\ldots,2n-1\}\setminus(X\cup\{x_1,x_3\})$. □

In the following, we use $\underbrace{x,y}$ to denote the labels x and y on two consecutive edges on a walk.

Theorem 2.1. *Given two vertices* w_0, w_{2n} *(not necessarily distinct) in* O_n $(n\ge 4)$ *with* $\lambda_1\notin[w_0]$ *and* $\lambda_2\notin[w_{2n}]$, *there exists a* $2n$*-special walk from* w_0 *to* w_{2n} *such that its first edge is labeled with* λ_1 *and the last edge is labeled with* λ_2.

Proof. We may assume that $[w_0]=\{x_1,x_2,\ldots,x_k\}\cup\{y_{k+1},y_{k+2},\ldots,y_{n-1}\}$, $[w_{2n}]=\{x_1,x_2,\ldots,x_k\}\cup\{z_{k+1},z_{k+2},\ldots,z_{n-1}\}$ and $[w_0]\cup[w_{2n}]\cup\{s_1,s_2,\ldots,s_{k+1}\}=\{1,\ldots,2n-1\}$, where $k\in\{0,1,\ldots,n-1\}$. By symmetry, we divide the proof into three cases.

Case 1. $\lambda_1\in\{z_{k+1},z_{k+2},\ldots,z_{n-1}\}$ and $\lambda_2\in\{y_{k+1},y_{k+2},\ldots,y_{n-1}\}$..

It is obvious that $k\ne n-1$. By symmetry, we may assume that $\lambda_1=z_{k+1}$ and $\lambda_2=y_{n-1}$.

Subcase 1.1. $k\le n-4$.

In the odd graph O_n, we can find a special walk from w_0 to w_{2n}, whose edges are labeled with $\underbrace{z_{k+1},y_{k+1}}$, $\underbrace{z_{k+2},y_{k+2}}$, $\ldots$, $\underbrace{z_{n-3},y_{n-3}}$, $\underbrace{s_1,y_{n-2}}$, $\underbrace{s_2,s_1}$, $\ldots$, $\underbrace{s_{k+1},s_k}$, $\underbrace{z_{n-2},s_{k+1}}$, $\underbrace{z_{n-1},y_{n-1}}$.

Subcase 1.2. $k=n-3$.

In the odd graph O_n, we can find a special walk from w_0 to w_{2n}, whose edges are labeled with $\underbrace{z_{n-2},x_1}$, $\underbrace{s_1,y_{n-2}}$, $\underbrace{s_2,s_1}$, $\underbrace{s_3,s_2}$, $\ldots$, $\underbrace{s_k,s_{k-1}}$, $\underbrace{x_1,s_k}$, $\underbrace{z_{n-1},y_{n-1}}$. (It needs n at least 4, otherwise x_1 does not exist.)