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On extremal hypergraphs for forests of tight paths

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In this paper, we investigate the maximal size of a k-uniform hypergraph containing no forests of tight paths, which extends the classical Erdős–Gallai Theorem for paths in graphs. Our results build on the results of Györi, Katona and Lemons, who considered the maximal size of a k-uniform hypergraph containing no single tight path.

1. Introduction

In this paper, we employ standard definitions and notation from hypergraph theory (see e.g., [1]). We use $e(\mathcal{G})$ to denote the size (i.e., the number of edges) of a hypergraph $\mathcal{G}$. By disjoint, we will always mean vertex disjoint.

Considering the extremal graphs without paths, Erdős and Gallai gave the following classical result.

Theorem 1.1. *(Erdős–Gallai [6]) Let G be a graph on n vertices containing no path of length k. Then $e(G) \le \frac{1}{2}(k-1)n$. Equality holds iff G is the disjoint union of complete graphs on k vertices.*

In [2], the authors determined the extremal results for forests of paths of the same length.

Theorem 1.2 ([2]). *For $n \geq 7k$, let G be a graph on n vertices containing no k disjoint paths of length 2. Then $e(G) \le \binom{k-1}{2} + (n-k+1)(k-1) + \lfloor \frac{n-k+1}{2} \rfloor$.*

This proves a conjecture of Gorgol [10]. For the paths of other length, Bushaw and Kettle proved the following result.

Theorem 1.3 (*[2]*). *For $k \geq 2$, $\ell \geq 4$ and $n \ge 2\ell + 2k\ell\left(\lceil \frac{\ell}{2} \rceil + 1\right)\binom{\ell}{\lfloor \frac{\ell}{2} \rfloor}$, let G be a graph on n vertices containing no k disjoint paths of length $\ell - 1$. Then*

$$e(G) \le \binom{k\lfloor \frac{\ell}{2} \rfloor - 1}{2} + \left(k\lfloor \frac{\ell}{2} \rfloor - 1\right)\left(n - k\lfloor \frac{\ell}{2} \rfloor + 1\right) + I_{\{\ell \text{ is odd}\}},$$

where $I_{\{\ell \text{ is odd}\}}$ is the indicator function of ℓ being odd.

Note that the bounds in above theorems are sharp. There are many results considering the cases not covered by Theorems 1.2 and 1.3, for the disjoint union of paths with the same length, see [4,5,19,23,24]. Then, the authors in [20] considered the extremal problem for the disjoint union of paths of different length.

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Theorem 1.4 ([20]). *Let G be a graph on n vertices containing no forest with paths of order $v_1, v_2, \ldots, v_k$. If at least one v_i is not 3, then for n sufficiently large,*

$$e(G) \le \left(\sum_{i=1}^{k} \left\lfloor \frac{v_i}{2} \right\rfloor - 1 \right) \left(n - \sum_{i=1}^{k} \left\lfloor \frac{v_i}{2} \right\rfloor + 1 \right) + \binom{\sum_{i=1}^{k} \lfloor \frac{v_i}{2} \rfloor - 1}{2} + c,$$

where $c = 1$ if all v_i are odd and $c = 0$ otherwise. Moreover, the extremal graph is unique.

There are numerous results related to the Turán numbers of paths and cycles, e.g. [13,18].

With a generalization of theorems above in mind, we now proceed to a discussion of the extremal problem on hyperpaths.

It is natural to consider the Erdős–Gallai Theorem on hypergraph version. There are several possible ways to define paths in hypergraphs. One such definition of paths in hypergraphs is due to Berge. A k-uniform *Berge path* of length ℓ is a family of ℓ distinct k-sets $F_1, \ldots, F_\ell$ and $\ell + 1$ distinct vertices $v_1, \ldots, v_{\ell+1}$ such that for each $1 \le i \le \ell$, F_i contains v_i and v_{i+1}. The authors in [11] found the extremal sizes of k-uniform hypergraphs avoiding Berge paths of length ℓ, and their results are sharp for infinitely many n.

Consider a further, more strict definition of a path that was first introduced by Katona and Kierstead in [16].

Definition 1.1. A tight path of length ℓ in a k-uniform hypergraph is a collection of $\ell + k - 1$ vertices $\{v_1, v_2, \ldots, v_{\ell+k-1}\}$ and ℓ edges $\{A_1, A_2, \ldots, A_\ell\}$ such that for each $1 \le i \le \ell$, $A_i = \{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$.

There are some other kinds of generalizations of paths of hypergraphs, here we present two of them.

Definition 1.2. A loose path of length ℓ in a k-uniform hypergraph is a collection of distinct edges $\{A_1, A_2, \ldots, A_\ell\}$ such that consecutive edges intersect in at least one element and nonconsecutive sets are disjoint.

Definition 1.3. A linear path of length ℓ in a k-uniform hypergraph is a collection of distinct edges $\{A_1, A_2, \ldots, A_\ell\}$ such that consecutive edges intersect in exactly one element and nonconsecutive sets are disjoint.

In [7], Füredi et al. determined the maximal size of a k-uniform hypergraphs without a loose path (for $k \ge 3$) or a linear path (for $k \ge 4$) of length ℓ. Kostochka et al. [15] also considered the maximal size of a k-uniform hypergraphs without a linear path of length ℓ for $k = 3$. Furthermore, Bushaw and Kettle [3] dealt with this extremal problem for forests of loose paths or linear paths.

There are numerous papers on the extremal problems of hypergraphs, the reader may refer to [8,12,14,21] for details.

For the tight paths, Györi et al. gave the following result.

Theorem 1.5 ([11]). *Let $\mathcal{H}$ be an extremal k-uniform hypergraph containing no tight path of length ℓ. Then*

$$(1 - o(1)) \frac{\ell - k + 1}{k} \binom{n}{k-1} \le |e(\mathcal{H})| \le (\ell - 1) \binom{n}{k-1}.$$

Very recently, Füredi et al. [9] improved the upper bound to $\frac{(\ell-1)(k-1)}{k}\binom{n}{k-1}$. Build on Theorem 1.5, we can get a quite exact result regarding hypergraphs avoiding disjoint union of tight paths. Our main result is as follows.

Theorem 1.6. *For $k \ge 3$ and $t \ge 1$, let $\mathcal{H}$ be an extremal k-uniform hypergraph containing no t disjoint tight paths of length $\ell_1, \ell_2, \ldots, \ell_t$ respectively, where $\ell_1 \le \ell_2 \le \ldots \le \ell_t$ and $\ell_1 \ge 2$. Let $\ell = \sum_{i=1}^{t} \ell_i$, then for sufficiently large n,*

$$(1 - o(1)) \frac{\ell_1 - 1}{k} \binom{n}{k-1} \le |e(\mathcal{H})| \le (\ell - 1) \binom{n}{k-1}.$$

Remark 1.1. If $t = 1$, i.e., the k-uniform hypergraph containing no single path of length ℓ. The lower bound given in Theorem 1.6 is better than the lower bound in Theorem 1.5.

2. Proof of Theorem 1.6

First we prove the lower bound. Actually, the construction of the lower bound is strongly connected to Steiner systems. For details, one can refer to the outstanding work of Keevash [17]. The following theorem given by Rödl will be useful to our construction.

Theorem 2.1 ([22]). *The packing number $m(n, \ell, s)$, i.e. the size of the largest ℓ-uniform family of subsets of an n-set such that every s-set is contained in at most 1 member of the family is $(1 + o(1)) \frac{\binom{n}{s}}{\binom{\ell}{s}}$.*

From Theorem 2.1, we can find a family $\mathcal{F}$ consisting of $(\ell_1 + k - 2)$-sets of an n-set, such that there is no $(k - 1)$-set which is contained in more than one element of the family, and