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Generalized tricobsthal and generalized tribonacci polynomials

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ABSTRACT

In this work, we will introduce generalized tribonacci and generalized tricobsthal polynomials. We introduce definitions, formulas for both families of polynomials and the Binet formulas, generating functions. We analyze special points for considered polynomials and present some of polynomials pictorially.

1. Introduction

Fibonacci polynomials are very well known from the literature (see for example [1,2]). They fulfill recurrence: $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$, where $f_1(x) = 1$ and $f_2(x) = x$ with property $f_n(1) = f_n$ is n-th Fibonacci number. Another interesting family of such kind polynomials is Jacobsthal polynomials (see for example [1,3–5]) or generalized Jacobsthal polynomials [6]. Jacobsthal polynomials are defined by recurrence relation: $\mathrm{J}_n(x) = \mathrm{J}_{n-1}(x) + x\mathrm{J}_{n-2}(x)$, where $\mathrm{J}_1(x) = 1$, $\mathrm{J}_2(x) = 1$ with property $\mathrm{J}_n(1) = f_n$.

The main idea of this paper is to investigate analog of tribonacci polynomials for Jacobsthal case (by analogy called tricobsthal polynomials) and generalized both cases i.e. introducing generalized tribonacci and generalized tricobsthal polynomials and describe some properties.

In Section 2 starting from tribonacci numbers we introduce tribonacci polynomials and generalize them. The Binet formula for considered polynomials is shown. In Section 3 in analogous way we define tricobsthal polynomials, generalized tricobsthal polynomials and the Binet formula for them. Section 4 is devoted to present properties of polynomials introduced in previous sections, such us explicit formula, generating functions, degree of polynomials and derivatives for them and others properties. Studying [1,6] we noticed that values of considered polynomials for $x = -1; 0; 1$ is interesting and Section 5 is devoted to this. In this section we also present some cases pictorially.

All polynomials considered in the paper are polynomials of real variable and real valued even if they are described by complex numbers.

2. Generalized tribonacci polynomials

Firstly let us start from tribonacci numbers [1] which are defined by

$$\mathrm{T}_n = \mathrm{T}_{n-1} + \mathrm{T}_{n-2} + \mathrm{T}_{n-3} \qquad \text{for } n \geqslant 4, \tag{1}$$

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with initial conditions $\mathrm{T}_1 = 1,\ \mathrm{T}_2 = 1$ and $\mathrm{T}_3 = 2$. First 10*th* tribonacci numbers are 1, 1, 2, 4, 7, 13, 24, 44, 81, 149. Next using [1] we present tribonacci polynomials defined by recurrence relation

$$t_n(x) = x^2 t_{n-1}(x) + x t_{n-2}(x) + t_{n-3}(x) \text{ for } n \geqslant 4,$$

with initial conditions

$$t_1(x) = 1,\ t_2(x) = x^2,\ t_3(x) = x^4 + x \tag{2}$$

and property $t_n(1) = \mathrm{T}_n$. First 10th tribonacci polynomials are:

$$\begin{aligned}
t_1(x) &= 1\\
t_2(x) &= x^2\\
t_3(x) &= x^4 + x\\
t_4(x) &= x^6 + 2x^3 + 1\\
t_5(x) &= x^8 + 3x^5 + 3x^2\\
t_6(x) &= x^{10} + 4x^7 + 6x^4 + 2x\\
t_7(x) &= x^{12} + 5x^9 + 10x^6 + 7x^3 + 1\\
t_8(x) &= x^{14} + 6x^{11} + 15x^8 + 16x^5 + 6x^2\\
t_9(x) &= x^{16} + 7x^{13} + 21x^{10} + 30x^7 + 19x^4 + 3x\\
t_{10}(x) &= x^{18} + 8x^{15} + 28x^{12} + 50x^9 + 45x^6 + 16x^3 + 1.
\end{aligned}$$

By analogy to general Fibonacci numbers (see for example [1]) let's consider conditions:

$$\begin{aligned}
T_1(x) &= a,\\
T_2(x) &= b_2x^2 + b_1x + b_0,\\
T_3(x) &= c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,
\end{aligned} \tag{3}$$

where b_2, c_1, c_4 positive integers and others parameters are nonnegative integers as initial conditions for tribonacci polynomials, then we have following definition:

Definition 1. Generalized tribonacci polynomials are defined by recurrence relation

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x) \text{ for } n \geqslant 4 \tag{4}$$

with initial conditions (3).

When $a = b_2 = c_1 = c_4 = 1$ and others parameters are zero, then one gets tribonacci polynomials with property $T_n(1) = t_n(1) = \mathrm{T}_n$.

When one considers any polynomials related with name Fibonacci should rewrite them in analog of the Binet formula and for considered in this section polynomials one gets following theorem:

Theorem 1. *The Binet formula for Generalized Tribonacci polynomials defined by (4) with initial conditions (3) is*

$$T_n(x) = C_{1,T}\alpha_T^{n-1} + C_{2,T}\beta_T^{n-1} + C_{3,T}\gamma_T^{n-1} \tag{5}$$

where n is positive integer,

$$\begin{aligned}
C_{1,T} &:= \frac{T_3(x) - (\gamma_T + \beta_T)T_2(x) + \gamma_T\beta_T T_1(x)}{(\alpha_T - \gamma_T)(\alpha_T - \beta_T)},\\
C_{2,T} &:= \frac{T_3(x) - (\gamma_T + \alpha_T)T_2(x) + \gamma_T\alpha_T T_1(x)}{(\beta_T - \gamma_T)(\beta_T - \alpha_T)},\\
C_{3,T} &:= \frac{T_3(x) - (\alpha_T + \beta_T)T_2(x) + \alpha_T\beta_T T_1(x)}{(\gamma_T - \alpha_T)(\gamma_T - \beta_T)}
\end{aligned}$$

and α_T, β_T, γ_T are different solutions of characteristic equation $y^3 - x^2y^2 - xy - 1 = 0$ of (4).

$$\begin{aligned}
\alpha_T &:= \frac{x^2}{3} - \frac{2^{\frac{1}{3}}(-3x - x^4)}{3\delta_T} + \frac{\delta_T}{3\cdot 2^{\frac{1}{3}}},\\
\beta_T &:= \frac{x^2}{3} + \frac{(1 + i\sqrt{3})(-3x - x^4)}{3\cdot 2^{\frac{2}{3}}\delta_T} - \frac{(1 - i\sqrt{3})\delta_T}{6\cdot 2^{\frac{1}{3}}},\\
\gamma_T &:= \frac{x^2}{3} + \frac{(1 - i\sqrt{3})(-3x - x^4)}{3\cdot 2^{\frac{2}{3}}\delta_T} - \frac{(1 + i\sqrt{3})\delta_T}{6\cdot 2^{\frac{1}{3}}}
\end{aligned} \tag{6}$$