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Cospectrality of graphs with respect to distance matrices

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ABSTRACT

The distance, distance Laplacian and distance signless Laplacian spectra of a connected graph G are the spectra of the distance, distance Laplacian and distance signless Laplacian matrices of G. Two graphs are said to be cospectral with respect to the distance (resp. distance Laplacian or distance signless Laplacian) matrix if they share the same distance (resp. distance Laplacian or distance signless Laplacian) spectrum. If a graph G does not share its spectrum with any other graph, we say G is determined by its spectrum. In this paper we are interested in the cospectrality with respect to the three distance matrices. First, we report on a numerical study in which we looked into the spectra of the distance, distance Laplacian and distance signless Laplacian matrices of all the connected graphs on up to 10 vertices. Then, we prove some theoretical results about what we can deduce about a graph from these spectra. Among other results we identify some of the graphs determined by their distance Laplacian or distance signless Laplacian spectra.

1. Introduction and definitions

We begin by recalling some definitions. In this paper, we consider only simple, undirected and finite graphs, i.e., undirected graphs on a finite number of vertices without multiple edges or loops. A graph is (usually) denoted by $G = G(V, E)$, where V is its vertex set and E is its edge set. The *order* of G is the number $n = |V|$ of its vertices and its *size* is the number $m = |E|$ of its edges.

As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by $K_{a,n-a}$ the complete bipartite graph and by K_n the complete graph, each on n vertices. The graph obtained from a star S_n, $n \geq 3$, by adding an edge is well defined and here denoted by S_n^+.

The *adjacency matrix* A of G is a 0–1 $n \times n$–matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. Denote by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the A–spectrum of G, i.e., the spectrum of the adjacency matrix of G, and assume that the eigenvalues are labeled such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For more results about the A–spectra of graphs, see the book [14].

The matrix $L = Deg - A$, where Deg is the diagonal matrix whose diagonal entries are the degrees in G, is called *the Laplacian* of G. Denote by $(\mu_1, \mu_2, \ldots, \mu_n)$ the L–spectrum of G, i.e., the spectrum of the Laplacian of G, and assume that the eigenvalues are labeled such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. For a survey about the Laplacian matrices of graphs see [34].

The matrix $Q = Deg + A$ is called *the signless Laplacian* of G. Denote by $(q_1, q_2, \ldots, q_n)$ the Q–spectrum of G, i.e., the spectrum of the signless Laplacian of G, and assume that the eigenvalues are labeled such that $q_1 \geq q_2 \geq \cdots \geq q_n$. For more details about the signless Laplacian of graphs see [15–17].

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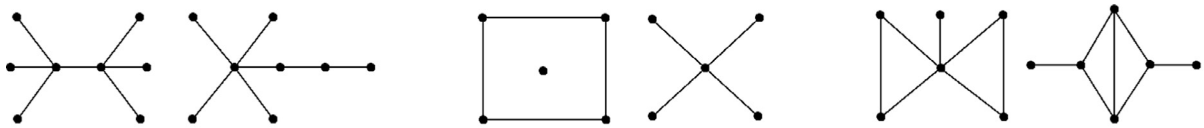

(a) Two A–cospectral trees. (b) Two A–cospectral graphs. (c) Two A–cospectral connected graphs.

Fig. 1. (a) Two A–cospectral trees. (b) Two A–cospectral graphs. (c) Two A–cospectral connected graphs.

Given two vertices u and v in a connected graph G, $d(u, v) = d_G(u, v)$ denotes the *distance* (the length of a shortest path) between u and v. The *Wiener index* $W(G)$ of a connected graph G is defined to be the sum of all distances in G, i.e.,

$$W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v).$$

The *transmission* $t(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e.,

$$t_v = t(v) = \sum_{u \in V} d(u, v).$$

A connected graph $G = (V, E)$ is said to be *k–transmission regular* if $t(v) = k$ for every vertex $v \in V$. The transmission regular graphs are exactly the *distance–balanced* graphs introduced in [28]. They are also called *self–median* graphs in [9].

The *distance matrix* $\mathcal{D}$ of a connected graph G is the matrix indexed by the vertices of G where $\mathcal{D}_{i,j} = d(v_i, v_j)$, and $d(v_i, v_j)$ denotes the distance between the vertices v_i and v_j. Let $(\partial_1, \partial_2, \ldots, \partial_n)$ denote the spectrum of $\mathcal{D}$. It is called the distance spectrum of the graph G. Assume that the distance eigenvalues are labeled such that $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. For a survey and references about the distance spectra of graphs, see [4].

Similarly to the (adjacency) Laplacian $L = Deg - A$, we defined in [3] the *distance Laplacian* of a connected graph G as the matrix $\mathcal{D}^L = Tr - \mathcal{D}$, where Tr denotes the diagonal matrix of the vertex transmissions in G. Let $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L)$ denote the spectrum of $\mathcal{D}^L$ and assume that the eigenvalues are labeled such that $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$. We call it the *distance Laplacian spectrum* of the graph G. Some properties of the distance Laplacian eigenvalues are discussed in [1]. In [35], Nath and Paul studied the second smallest distance Laplacian eigenvalue ∂_{n-1}^L and characterized some families of graphs for which $\partial_{n-1}^L = n + 1$. They [35] also studied the distance Laplacian spectrum of the path P_n.

Also in [3], and similarly to the (adjacency) signless Laplacian $L = Deg + A$, we introduced the *distance signless Laplacian* of a connected graph G to be $\mathcal{D}^Q = Tr + \mathcal{D}$. Let $(\partial_1^Q, \partial_2^Q, \ldots, \partial_n^Q)$ denote the spectrum of $\mathcal{D}^Q$ and assume that the eigenvalues are labeled such that $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$. We call it the *distance signless Laplacian spectrum* of the graph G.

For a given real number x and a matrix M, we denote $\mu_M(x)$ the multiplicity of x as an eigenvalue of M. Evidently, $\mu_M(x) = 0$ whenever x does not belong to the spectrum of M.

Graphs with the same spectrum with respect to an associated matrix M are called *cospectral* graphs with respect to M, or *M–cospectral graphs*. Two M–cospectral non–isomorphic graphs G and H are called *M–cospectral mates* or *M–mates*. If one considers more than one matrix associated to graphs, say $M_1, M_2, \ldots, M_k$, then two graphs are said to be $(M_1, M_2, \ldots, M_k)$-mates if they are cospectral with respect to all the matrices $M_1, M_2, \ldots, M_k$ simultaneously.

In the next section, and after a brief review on the cospectrality with respect to A, L and Q, we report on a numerical study in which we looked into the spectra of the distance, distance Laplacian and distance signless Laplacian matrices of all the connected graphs on up to 10 vertices. To achieve our objective we generated the desired graphs using *Nauty* (a computer program for generating graphs available at http://www.cs.anu.edu.au/~bdm/nauty/) and then calculated the different spectra using the third version of AutoGraphiX (AGX III) [10].

In the last section, we prove some theoretical results about what we can deduce about a graph from these spectra. Among others results we identify some of the graphs determined by their distance Laplacian or distance signless Laplacian spectra.

2. Experiments

The question "Which graphs are determined by their A–spectrum?" raised by Günthard and Primas [24] in 1956 in a paper relating spectral theory of graphs and Hückel's theory from chemistry. Cospectrality plays an important role in isomorphism theory. Actually, it is not yet known if testing isomorphism of two graphs is a hard problem or not, while determining whether two graphs are cospectral can be done in polynomial time. Thus checking isomorphism is done among cospectral graphs only. It was conjectured [24] that there are no A–cospectral mates. A year later, the conjecture was refuted by Collatz and Sinogowitz [12] giving two A–cospectral mates, which are in fact, the smallest such trees (see Fig. 1 (a)). For the class of general graphs, the A–cospectral mates with the smallest order, first given by Cvetković [13], are illustrated in Fig. 1 (b). The A–cospectral mates with smallest order among connected graphs, first given by [6], are illustrated in Fig. 1 (c).

Several constructions of A–cospectral mates were proposed in the literature. The first infinite family of pairs of A–cospectral mates, among trees, was constructed by Schwenk [37]. For more construction methods see for example