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Some sufficient conditions on k-connected graphs☆

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ABSTRACT

A connected graph G is said to be k-connected if it has more than k vertices and remains connected whenever fewer than k vertices are removed. Feng et al. (2017) presented sufficient conditions based on spectral radius for a graph to be k-connected, k-edge-connected, k-Hamiltonian, k-edge-Hamiltonian, β-deficient and k-path-coverable. In this paper, we present some further sufficient conditions for a graph to be k-connected in terms of signless Laplacian spectral radius, distance spectral radius, distance signless Laplacian spectral radius of G and Wiener index, Harary index of its complement.

1. Introduction

All graphs considered here are undirected graphs without loops and multiple edges. We use the book of Bondy and Murty [3] for terminology and notation not defined here. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. We denote by $d_G(v)$ (or simply $d(v)$) the degree of v, δ the minimum degree of G. Let K_n denote a complete graph and $K_{s,t}$ denote a complete bipartite graph. For two disjoint graphs G and H, we use $G+H$ and $G \vee H$ to denote the union and the join of G and H, respectively. The union of k disjoint copies of the same graph G is denoted by kG. Let $K_{n-1}+v$ denote the complete graph on $n-1$ vertices together with an isolated vertex. The complement $\overline{G}$ of G is the graph with the vertex set $V(G)$, and two vertices $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A connected graph G is said to be k-connected (or k-vertex connected) if it has more than k vertices and remains connected whenever fewer than k vertices are removed. The connectivity $\kappa(G)$ of G is the maximum value of k for which G is k-connected.

An integer sequence $\pi = (d_1 \le d_2 \le \cdots \le d_n)$ is called graphical if there exists a graph G having π as its vertex degree sequence, in that case, G is called a realization of π. If P is a graph property, such as hamiltonian or k-connected, we call a graphical sequence π forcibly P if every realization of π has property P. Historically, the vertex degrees of a graph have been used to provide sufficient conditions for the graph to have certain properties, such as Hamiltonicity or k-connectedness.

The adjacency matrix of G is $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$, denoted by $\mu(G)$, is called to be the spectral radius of G. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G and the matrix $Q(G) = D(G) + A(G)$ is the signless Laplacian matrix of G, where $D(G)$ is the degree diagonal matrix of G. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called to be the signless Laplacian spectral radius of G.

The distance between u and v in G is the length of a shortest path from u to v. We denote it by $d_G(u,v)$. The transmission $Tr(u) = Tr_G(u)$ of a vertex u is defined to be the sum of distances from u to all other vertices in G,

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i.e., $Tr(u) = \sum_{v \in V(G)} d_G(u, v)$. If $Tr(u)$ is a constant for each $u \in V(G)$, then the graph G is said to be transmission regular. The transmission of G, denoted by $\sigma(G)$, is the sum of distances between every pair of vertices of G. Obviously, we have $\sigma(G) = \frac{1}{2} \sum_{u \in V(G)} Tr(u)$. The distance matrix of G, denoted by $\mathcal{D}(G)$, is a symmetric real matrix with (i, j)-entry being $d_G(v_i, v_j)$. The largest eigenvalue of $\mathcal{D}(G)$, denoted by $\rho(G)$, is called to be the distance spectral radius of G. Let $Tr(G)$ be the diagonal matrix of the vertex transmissions in G. The matrix $Q_D(G) = Tr(G) + \mathcal{D}(G)$ is the distance signless Laplacian matrix of graph G. The largest eigenvalue of $Q_D(G)$, denoted by $\rho_D(G)$, is called to be the distance signless Laplacian spectral radius of G.

In [4], the following problem was proposed:

Problem: Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radii of the graphs of $\mathcal{G}$, and characterize the graphs for which the maximal spectral radius is attained.

Motivated by this, recently, there are many reasonable sufficient or necessary spectral conditions that were given for a graph to have some property P. Fiedler and Nikiforov [10] firstly gave tight sufficient conditions for the existence of Hamilton paths and cycles in terms of the spectral radius of graphs or the complement of graphs. The other related results on this aspect can be found in [8,21–25,30,32–34]. In [7], Feng et al. presented sufficient conditions based on spectral radius for a graph to be k-connected, k-edge-connected, k-Hamiltonian, k-edge-Hamiltonian, β-deficient and k-path-coverable. In this paper, we give further spectral conditions on signless Laplacian spectral radius, distance spectral radius, distance signless Laplacian spectral radius of G for the graph to be k-connected.

The Wiener index $W(G)$ of a connected graph G is defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$. The Wiener index was introduced in 1947 by Wiener [27], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. Obviously, $W(G) = \sigma(G)$. Throughout the years, the Wiener index has been extensively studied. In [27], Wiener also introduced another topological index, called Wiener polarity index $W_p(G)$, which is the number of unordered pairs of vertices that are at distance 3 in G. It also has attracted much attention in recent years. For more details, we can refer to [5,11,16].

The Harary index $H(G)$ of a graph G has been introduced independently by Ivanciuc et al. [15] and Plavšić et al. [26] in 1993 for the characterization of molecular graphs. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index of a graph is defined as $H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u,v)}$. We use $\hat{Tr}(v_i)$ to denote $\sum_{v_j \in V(G)} \frac{1}{d_G(v_i,v_j)}$. Then $H(G) = \frac{1}{2} \sum_{i=1}^{n} \hat{Tr}(v_i)$.

Recently, some sufficient conditions on Wiener index and Harary index for a graph to be Hamiltonian and traceable have been given, see [12,13,17–20,29,31]. In [9], Feng et al. gave some sufficient conditions on Wiener index and Harary index of G for the graph to be k-connected, k-edge-connected, k-Hamiltonian, k-edge-Hamiltonian, β-deficient and k-path-coverable. In this paper, we also give some sufficient conditions in terms of Wiener index, Harary index of $\overline{G}$ for the graph to be k-connected.

2. Main results

Lemma 2.1 [2]**.** *Let G be a connected graph and $\pi = (d_1 \le d_2 \le \cdots \le d_n)$ be a graphical sequence. Suppose $n \ge 2$ and $1 \le k \le n - 1$. If*

$$d_i \le i + k - 2 \Rightarrow d_{n-k+1} \ge n - i, \ \text{for } 1 \le i \le \frac{1}{2}(n - k + 1),$$

then π is forcibly k-connected.

The following Lemma 2.2 is the claim in the proof of Theorem 3.1 in [7], here we give a different proof of it.

Lemma 2.2 [7]**.** *Let G be a graph of order $n \ge k + 1$ with m edges. If*

$$m \ge \frac{n^2 - 3n}{2} + k,$$

then G is k-connected unless $G = K_{k-1} \vee (K_1 + K_{n-k})$.

Proof. Suppose that G is not k-connected and has the degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \le d_2 \le \cdots \le d_n$, $n \ge k + 1$. By Lemma 2.1, there is an integer $1 \le i \le \frac{1}{2}(n - k + 1)$ such that $d_i \le i + k - 2$ and $d_{n-k+1} \le n - i - 1$. Thus,

$$\begin{aligned} m &= \frac{1}{2} \sum_{s=1}^{n} d_s \\ &= \frac{1}{2} \left(\sum_{s=1}^{i} d_s + \sum_{s=i+1}^{n-k+1} d_s + \sum_{s=n-k+2}^{n} d_s \right) \\ &\le \frac{1}{2} [i(i + k - 2) + (n - k - i + 1)(n - i - 1) + (k - 1)(n - 1)] \\ &= \frac{n^2 - 3n}{2} + k - (i - 1)(n - i - k). \end{aligned}$$