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On bivariate classical orthogonal polynomials

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ABSTRACT

We deduce new characterizations of bivariate classical orthogonal polynomials associated with a quasi-definite moment functional, and we revise old properties for these polynomials. More precisely, new characterizations of classical bivariate orthogonal polynomials satisfying a diagonal Pearson-type equation are proved: they are solutions of two separate partial differential equations one for every partial derivative, their partial derivatives are again orthogonal, and every vector polynomial can be expressed in terms of its partial derivatives by means of a linear relation involving only three terms of consecutive total degree. Moreover, we study general solutions of the matrix second order partial differential equation satisfied by classical orthogonal polynomials, and we deduce the explicit expressions for the matrix coefficients of the structure relation. Finally, some illustrative examples are given.

1. Introduction

In 1929, Bochner [1] characterized classical orthogonal polynomials in one variable, namely, Hermite, Laguerre, Jacobi, and Bessel, as the only families of orthogonal polynomials that are solutions of the second order linear differential equation

$$\phi(x)\, y'' + \psi(x)\, y' = \lambda_n\, y,$$

where $\phi(x)$ and $\psi(x)$ are fixed polynomials of degree ≤ 2 and exactly 1, respectively, and λ_n is a real number depending on the degree of the polynomial solution. Moreover, classical orthogonal polynomials in one variable are also characterized as those sequences $\{p_n\}_{n\geq 0}$ of orthogonal polynomials associated with a quasi-definite moment functional $\mathbf{u}$ satisfying the Pearson-type equation

$$D(\phi\, \mathbf{u}) = \psi\, \mathbf{u},$$

where D denotes the distributional derivative operator (see, for instance, [2], among others). Moreover, the authors in [2,3] showed that a quasi-definite moment functional is classical if and only if its moments $\{\mu_n\}_{n\geq 0}$ satisfy a three term recurrence relation

$$r_n\, \mu_{n+1} + s_n\, \mu_n + n\, \phi(0)\, \mu_{n-1} = 0, \quad n \geq 0, \mu_{-1} = 0, \mu_0 = 1, \tag{1}$$

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where $r_n \neq 0$ for $n \geq 0$, and r_n, s_n are given in terms of ϕ and ψ.

Orthogonal polynomials in two variables are studied as the natural generalization of orthogonal polynomials in one variable. In 1967, Krall and Sheffer [4] extended the concept of classical sequence of orthogonal polynomials to the bivariate case, and defined classical orthogonal polynomials as the polynomial solutions of the second order partial linear differential equation

$$\begin{aligned}(\alpha x^2 + d_1 x + e_1 y + f_1) p_{xx} + (2\alpha x y + d_2 x + e_2 y + f_2) p_{xy} \\ +(\alpha y^2 + d_3 x + e_3 y + f_3) p_{yy} + (\delta x + h_1) p_x + (\delta y + h_2) p_y = \lambda_n p,\end{aligned} \tag{2}$$

where they impose $\lambda_n = n[(n-1)\alpha + \delta] \neq 0,$ and $n \geq 0$ is the total degree of the polynomial solution. The special shape of the polynomial coefficients comes from the fact that all orthogonal polynomials of the same total degree have to be solutions of Eq. (2).

In 1975, Koornwinder [5] studied examples of two variable orthogonal polynomials introducing seven classes of orthogonal polynomials which he considered bivariate analogues of the Jacobi polynomials. Some of these examples are classical as defined by Krall and Sheffer, such as orthogonal polynomials on the unit disk or on the simplex with respect to some specific weight functions. However, some of the Koornwinder classes are not classical according to the Krall and Sheffer definition. In his paper, Koornwinder used a very interesting tool, previously introduced by Agahanov [6], to construct orthogonal polynomials in two variables from orthogonal polynomials in one variable. Recently, this method has been extended to the quasi-definite definite case in [7].

Some interesting questions appear when the extension of the usual characterizations for classical bivariate orthogonal polynomials are studied. Firstly, Lyskova [8], tried to extend the well known property stating that the derivatives of univariate classical orthogonal polynomials are again classical orthogonal polynomials of the same type. This author studied conditions on the polynomial coefficients in (2) in such a way that the partial derivatives of the orthogonal polynomial solutions satisfy a partial differential equation of the same type. He deduced that $e_1 = e_3 = 0,$ that is, the polynomial coefficients of the partial derivatives in x depend only on x, and the polynomial coefficients of the partial derivatives in y depend only on y. The set of such a type of bivariate classical orthogonal polynomials was called the *basic class* or *Lyskova class*.

In [9,10], using the partial differential Eq. (2), the extension of the well-known properties as the *structure relation* for the bivariate orthogonal polynomials, the *matrix Pearson-type* equation for the moment functional, and a Rodrigues-type formula were deduced. This kind of properties also appeared in the Suetin book [11].

In [12,13], the authors extended the concept of classical orthogonal polynomials in two variables to a wider framework, which includes those of the Krall and Sheffer definition and the seven classes introduced by Koornwinder. This is achieved by means of the vector notation for polynomials in several variables. In this extended definition, classical orthogonal polynomials arranged in a vector form satisfy a matrix second order partial differential equation analogous to (2), but whose coefficients are polynomials of limited degrees 2 and 1, respectively, without any *a priori* constraint on their shape. With this new perspective, the authors proved the usual characterizations from a matrix point of view: the *structure relation for the bivariate vector orthogonal polynomials*, the *matrix Pearson-type* equation for the moment functional, and the *orthogonality of the gradients* of the vector of orthogonal polynomials. By using this new definition for the classical character, they extended another properties: a Rodrigues-type formula [14] or the Lyskova class [15]. However, there exist several properties of bivariate classical orthogonal polynomials that can be still studied.

In this paper, we will give new characterizations for classical orthogonal polynomials in two variables, and we obtain explicit expressions for the matrix coefficients. In particular, we deduce that the moments of a bivariate classical moment functional arranged in a vector form satisfy a matrix three term relation similar to (1) that characterizes them. We also give three new characterizations for classical moment functionals that satisfy a diagonal matrix Pearson-type equation. Although some of these characterizations were deduced in [16] in the framework of the Lyskova class, we only need the diagonal character of the matrix Pearson-type equation. In addition, we study properties of existence and uniqueness of the polynomial solutions for the matrix partial differential equation (the matrix extension of (2)).

The paper is organized as follows. In Section 2, we introduce the basic definitions, tools, and background we will need in the sequel. Then, we prove that a moment functional is classical if and only if its moments satisfy two matrix three term relations in Section 3. Next Section is focused on three characterizations for classical moment functionals that satisfy a diagonal matrix Pearson-type equation. In Section 5, we study the existence and uniqueness of polynomial solutions of the matrix second order partial differential equation that characterize classical orthogonal polynomials in the extended sense, and in Section 6, from the explicit expressions for the coefficient polynomials in the matrix partial second order differential equation, we deduce explicit expressions for the matrix coefficients in the characterizations of bivariate classical orthogonal polynomials. Finally, last Section is devoted to provide several interesting examples to illustrate the results obtained in this paper.

2. Basic facts

For each $n \geq 0$, let