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A structure-preserving split finite element discretization of the split wave equations

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ABSTRACT

We introduce a new finite element (FE) discretization framework applicable for covariant split equations. The introduction of additional differential forms (DF) that form pairs with the original ones permits the splitting of the equations into topological momentum and continuity equations and metric-dependent closure equations that apply the Hodge-star operator. Our discretization framework conserves this geometrical structure and provides for all DFs proper FE spaces such that the differential operators (here gradient and divergence) hold in strong form. We introduce lowest possible order discretizations of the split 1D wave equations, in which the discrete momentum and continuity equations follow by trivial projections onto piecewise constant FE spaces, omitting partial integrations. Approximating the Hodge-star by nontrivial Galerkin projections (GP), the two discrete metric equations follow by projections onto either the piecewise constant (GP0) or piecewise linear (GP1) space.

Out of the four possible realizations, our framework gives us three schemes with significantly different behavior. The split scheme using twice GP1 is unstable and shares the dispersion relation with the P1–P1 FE scheme that approximates both variables by piecewise linear spaces (P1). The split schemes that apply a mixture of GP1 and GP0 share the dispersion relation with the stable P1–P0 FE scheme that applies piecewise linear and piecewise constant (P0) spaces. However, the split schemes exhibit second order convergence for both quantities of interest. For the split scheme applying twice GP0, we are not aware of a corresponding standard formulation to compare with. Though it does not provide a satisfactory approximation of the dispersion relation as short waves are propagated much too fast, the discovery of the new scheme illustrates the potential of our discretization framework as a toolbox to study and find FE schemes by new combinations of FE spaces.

1. Introduction

The Finite Element (FE) method provides a powerful framework to discretize partial differential equations (PDEs) and includes methods to prove the discrete models' convergence, stability, and accuracy properties (see e.g. [11,25]). By offering

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flexibility in the choice of computational (unstructured, h/p-adapted) meshes (cf. [8]) while providing an approximation of the continuous PDEs with the required order of accuracy, FE discretizations are nowadays appreciated in all research areas that apply numerical modeling.

Discretizations using finite element methods provide one important advantage over other methods: starting from a variational formulation the discretization follows simply by substituting discrete function spaces for the continuous spaces (Galerkin methods) while the differential operators remain unchanged. There exist a large variety of different suitable FE spaces to choose from. However, not all choices lead to well-behaved schemes. In particular mixed FE schemes suffer from this problem, where different variables of the PDE system are represented by different FE spaces. In such schemes, certain combinations of FE spaces lead to instabilities that exhibit spurious modes, rendering the solutions useless, in particular when studying nonlinear phenomena. A famous example for an unstable scheme is given by an approximation of both velocity and height fields of the 1D shallow-water equations by piecewise linear functions (cf. [28] and Section 3), where it is well known that equal order FE pairs are always unstable [11,13].

In order to avoid unsuitable choices, the Finite Element Exterior Calculus (FEEC) method [2,3] provides means for choosing a suitable pair of FE spaces that is guaranteed to lead to a stable mixed discretization. In particular, FEEC puts geometrical constraints on the FE spaces such that geometric properties, like the Helmholtz decomposition of vector fields, are preserved in the discrete case. As a result, FEEC pairs of spaces always satisfy the inf-sup condition [1] while combinations of FE spaces that are *not* stable are ruled out. For the above mentioned 1D wave equations, approximating the velocity with piecewise linear and the height field with piecewise constant spaces satisfies the requirements of FEEC and gives indeed a stable scheme (cf. again [28] and Section 3).

Although providing a very general mathematical framework, naturally there are issues for which FEEC yields no satisfying answers. Let us consider, for instance, problems in geophysical fluid dynamics (GFD), in which an additional Coriolis term in the equations introduces effects caused by the Earth's rotation [27]. For an atmosphere in rest, the Coriolis force that depends on the velocity and the gradient of the pressure (or height) are in *geostrophic balance*. To maintain this balance in the discretization, the pressure (or height) field should be represented discretely at one order of consistency higher than that of the velocity field. Unfortunately, this contradicts the requirement imposed by FEEC on this FE pair. In [13], this issue could be resolved by applying a combination of FE and Discontinuous Galerkin (DG) spaces.

Moreover, in order to meet the regularity requirement of the chosen FE pairs, FEEC requires the PDEs to be written in weak variational form, in which partial integration has been performed. As pointed out recently in [20], the conventional mixed (weak) form of the equations causes certain operators, such as the co-derivative, to be non-local (global) operators. As a consequence, such FEEC methods are not locally volume preserving, which reduces the quality of the local representation of the quantities of interest (cf. [20]).

In this manuscript, our main goal is to introduce a FE discretization framework that provides an alternative methodology to avoid mentioned unsuitable FE choices with GFD in mind. More specifically, we develop a framework that applies two FEEC pairs instead of one, therefore providing a larger variety of different combinations of FE spaces, in which both derivatives and co-derivatives are local operators. This framework is based on formulating the PDEs in split form, as introduced in [4,5] for the GFD equations. The split equations consist of a topological and metric part while employing straight and twisted differential forms to adequately model the physical quantities of interest. The FE discretization framework translates this geometrical structure from the continuous to similarly structured discrete equations (cf. Section 2).

Our approach shares some basic ideas with other structure-preserving discretization methods (see e.g. [6,7,17,18]), but in particular with *mimetic discretizations* (see e.g. [9,10,12,16,24], and [26] for a historical overview). In the latter methods, the PDEs are also formulated by differential forms and a clear distinction between purely topological and metric terms is achieved. Applying algebraic topology as the discrete counterpart to differential geometry, the discrete equations mimic the underlying geometrical structure and are therefore denoted as structure-preserving (cf. [19]). Similar ideas of distinguishing between metric-dependent and metric-free terms in a GFD related context can also be found in [15] introducing FEEC discretizations of the nonlinear rotating shallow-water equations. In spite of these similarities, none of the schemes associate a proper FE space to each variable, as suggested by our framework.

For the sake of a clear exposition, we focus on a simple example, the split 1D linear shallow-water set of equations. Extending our framework to treat also more practically relevant equations, such as the nonlinear rotating shallow-water equations, is the subject of ongoing and future work. By investigating structure-preserving methods that apply lowest order (piecewise linear and constant) FE spaces to keep computational costs low, we address the requirements of GFD in developing schemes that satisfy first principle conservation laws (i.e mass and momentum conservation) and that are suited for simulations with integration times in the order of years and longer.

We structure the manuscript as follows. In Section 2, we introduce the split set of 1D wave equations and motivate the use of the split form of the equations as the fundamental formulation for their discretization. Recalling in Section 3 two low-order mixed FE schemes, namely the unstable P1–P1 and the stable P1–P0 pairs, we introduce in Section 4 the new discretization framework, referred to as the *split FE method*. We suggest a solving algorithm and present the schemes' discrete dispersion relations. Comparing them if possible with the conventional mixed schemes, we perform in Section 5 numerical simulations to investigate conservation behavior, convergence rates, and accuracy of all schemes. Finally in Section 6, we draw conclusions and provide an outlook for ongoing and future work.