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A Galerkin energy-preserving method for two dimensional nonlinear Schrödinger equation

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ABSTRACT

In this paper, a Galerkin energy-preserving scheme is proposed for solving nonlinear Schrödinger equation in two dimensions. The nonlinear Schrödinger equation is first rewritten as an infinite-dimensional Hamiltonian system. Following the method of lines, the spatial derivatives of the nonlinear Schrödinger equation are approximated with the aid of the Galerkin methods. The resulting ordinary differential equations can be cast into a canonical Hamiltonian system. A fully-discretized scheme is then devised by considering an average vector field method in time. Moreover, based on the fast Fourier transform and the matrix diagonalization method, a fast solver is developed to solving the resulting algebraic equations. Finally, the proposed scheme is employed to capture the blow-up phenomena of the nonlinear Schrödinger equation.

1. Introduction

In this paper, we consider the following cubic nonlinear Schrödinger (NLS) equation in two dimensions

$$\mathrm{i}u_t + \Delta u + \alpha|u|^2 u = 0, \quad (x, y) \in \Omega, \quad 0 < t \le T, \tag{1.1}$$

with periodic bounding conditions

$$u(x, y, t) = u(x + L_x, y, t), \tag{1.2}$$

$$u(x, y, t) = u(x, y + L_y, t), \ \ (x, y) \in \Omega, \ \ 0 \le t \le T, \tag{1.3}$$

and initial condition

$$u(x, y, 0) = u_0(x, y), \ \ (x, y) \in \Omega, \tag{1.4}$$

where $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplacian operator, $\mathrm{i} = \sqrt{-1}$, $\alpha \ne 0$ is a real constant, $\Omega = \Omega_x \times \Omega_y = [x_L, x_R] \times [y_L, y_R]$, $L_x = x_R - x_L$, and $L_y = y_R - y_L$. Eq. (1.1) is focusing for $\alpha > 0$, and defocusing for $\alpha < 0$, which plays a key role in modeling different physical phenomena, (e.g., see Refs. [14,25] and references therein). Known strategies to solve the NLS equation

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numerically include the time-splitting pseudo-spectral methods (e.g., see Refs. [4,19,32,40]), the finite difference methods (e.g., see Refs. [18,29,39,43]), the finite element methods (e.g., see Refs. [2,28]) and the meshless methods (e.g., see Refs. [12,13]) etc.

Let $u(x, y, t) = p(x, y, t) + q(x, y, t)\mathrm{i}$. Eq. (1.1) is equivalent to

$$p_t + \Delta q + \alpha(p^2 + q^2)q = 0, \tag{1.5}$$

$$q_t - \Delta p - \alpha(p^2 + q^2)p = 0. \tag{1.6}$$

Eqs. (1.5) and (1.6) can be comprised to an infinite-dimensional Hamiltonian system [29]

$$\frac{d\boldsymbol{z}}{dt} = \boldsymbol{S}\frac{\delta\mathcal{H}(\boldsymbol{z})}{\delta\boldsymbol{z}}, \tag{1.7}$$

where $\boldsymbol{z} = (p, q)^{\mathsf{T}}$,

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the Hamiltonian

$$\mathcal{H}(\boldsymbol{z}) = \int_\Omega \frac{1}{2}\left[p_x^2 + p_y^2 + q_x^2 + q_y^2 - \frac{\alpha}{2}(p^2 + q^2)^2\right]dxdy.$$

In fact, we can deduce from (1.7) that

$$\frac{d}{dt}\mathcal{H}(\boldsymbol{z}(t)) = \frac{\delta\mathcal{H}(\boldsymbol{z})}{\delta\boldsymbol{z}}^T\frac{d\boldsymbol{z}}{dt} = \frac{\delta\mathcal{H}(\boldsymbol{z})}{\delta\boldsymbol{z}}^T\boldsymbol{S}\frac{\delta\mathcal{H}(\boldsymbol{z})}{\delta\boldsymbol{z}} = 0,$$

which implies that the system (1.7) admits the conservation of the Hamiltonian energy.

In the past several decades, the theory and the methods for Hamiltonian ordinary differential equations (ODEs) are well-developed (e.g., see Refs. [7,9,10,16,17,23,24,26,27] and references therein). However, due to the fact that the phase space goes from finite to infinite dimension for Hamiltonian partial differential equations (PDEs), the major difficulties arise immediately when the methods are extended from Hamiltonian ODEs to Hamiltonian PDEs. A popular method to treat Hamiltonian PDEs is the so-called method of lines which first discretizates the PDEs in space resulting in a large system of Hamiltonian ODEs. The resulting ODEs are then integrated by the appropriate methods in time. However, how to guarantee the semi-discretization of Hamiltonian PDEs is still a canonical Hamiltonian system is challenging. Fortunately, it has shown that the finite difference methods [33], the pseudo-spectral methods [6,11], and the wavelet collocation method [45] etc have played an important role in methods of lines. However, comparing with the conventional methods, there have few approaches taking advantages of the weak formulation of the Hamiltonian system in the literature to our knowledge. Thus, there are a wide range of interests to introduce the Galerkin method for the Hamiltonian PDEs.

Recently, based on the weak formulations of the Hamiltonian system, the Galerkin finite element method [31] and the Legendre spectral element method [30] were employed to solve the NLS equation in one dimension. It showed that they could preserve satisfactorily the Hamiltonian structure of the original Hamiltonian PDEs and required weaker smoothness of the solution than the finite difference method (see Ref. [31]). But, the Galerkin finite element methods for Hamiltonian systems in two dimensional case are comparatively less explored. This motivates our study to introduce a Galerkin method for the two dimensional NLS equation under the framework of the Hamiltonian PDEs. The principal difficulty in two dimensional case is how to choose appropriate basis functions and partitions of Ω, so that the resulting ODEs are Hamiltonian. One of our aims in this paper is to propose a Galerkin method which can preserve the Hamiltonian structure of the NLS equation in two dimensions.

As is well-known that energy conservation law has played an important role in the theory of solitons of the NLS equation. Moreover, Fei et al. have pointed out that the nonconservative schemes of the NLS equation may easily show nonlinear blow-up [15]. Thus, devising the numerical schemes, which can preserve the energy conservation law of NLS equation, attracts a lot of interest (e.g., see Refs. [3,5,20,21,41–43]). However, there are few energy-preserving schemes on NLS equation by the Galerkin method to our knowledge. Thus, another aim in this paper is to propose an energy-preserving Galerkin method for the NLS equation in two dimensions.

Altogether, our main objectives in this paper are as follows:

1. A Galerkin method, which can guarantee the semi-discretization of the NLS equation is still a canonical Hamiltonian system, is introduced. Moreover, the corresponding structure matrix and the Hamiltonian function are obtained explicitly.
2. An energy-preserving Galerkin method is proposed for the NLS equation by using the average vector field (AVF) method [37] in time and the Galerkin method in space.
3. With the use of the matrix diagonalization method [38] and the fast Fourier transform (FFT), a fast solver is developed for solving the proposed scheme.

The paper is organized as follows. In Section 2, a finite-dimensional canonical Hamiltonian formulation of the NLS equation is obtained by using the Galerkin method. A new energy-preserving Galerkin method is proposed in Section 3. A fast solver and some numerical experiments are presented in Section. 4. Finally, some concise conclusions are drawn in Section 5.