Applied Mathematics and Computation 324 (2018) 51–58

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A higher-order convolution for Bernoulli polynomials of the second kind

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ARTICLE INFO

MSC:
11B83
11B68
05A15

Keywords:
Bernoulli numbers of the second kind
Bernoulli polynomials of the second kind
Cauchy numbers
Cauchy polynomials
Convolution identity

ABSTRACT

In this paper, we perform a further investigation for the Bernoulli polynomials of the second kind. By making use of the generating function methods and summation transform techniques, we establish a higher-order convolution identity for the Bernoulli polynomials of the second kind. We also present some illustrative special cases as well as immediate consequences of the main result.

1. Introduction

Throughout this paper, we always denote by $\binom{\alpha}{k}$ the binomial coefficients given for complex number α and non-negative integer k,

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!} \quad (k \geq 1), \tag{1.1}$$

and we also denote by $\binom{n}{r_1,\dots,r_k}$ the multinomial coefficients given for positive integer k and non-negative integers $n, r_1, \dots, r_k$,

$$\binom{n}{r_1,\dots,r_k} = \frac{n!}{r_1!\dots r_k!}. \tag{1.2}$$

We recall here the higher order Bernoulli polynomials $B_n^{(d)}(x)$ defined for non-negative integer n and positive integer d by the following generating function (see, e.g., [14,16,28]):

$$\left(\frac{t}{e^t-1}\right)^d e^{xt} = \sum_{n=0}^{\infty} B_n^{(d)}(x)\frac{t^n}{n!} \quad (|t| < 2\pi). \tag{1.3}$$

In particular, the case $d = 1$ in (1.3) gives the classical Bernoulli polynomials $B_n(x) = B_n^{(1)}(x)$ and the classical Bernoulli numbers $B_n = B_n^{(1)}(0)$, respectively.

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https://doi.org/10.1016/j.amc.2017.12.014
0096-3003/

In the year 2007, by using some convolution identities for Stirling numbers of the second kind and for sums of powers of integers, both involving the classical Bernoulli numbers, Agoh and Dilcher [1] obtained an explicit formula for

$$(B_k+B_m)^n=\sum_{i=0}^{n}\binom{n}{i}B_{k+i}B_{m+n-i}\quad (k,m,n\geq 0), \tag{1.4}$$

by virtue of which they deduced some surprising and unusual convolution identities including the famous Euler's formula for the classical Bernoulli numbers. See also [2,8,27] for different proofs for (1.4) and [12,13] for the expression of

$$(B_k(x)+B_m(y))^n=\sum_{i=0}^{n}\binom{n}{i}B_{k+i}(x)B_{m+n-i}(y)\quad (k,m,n\geq 0). \tag{1.5}$$

As a further extension of (1.4), Agoh and Dilcher [3] studied the expressions for the following general cases when $k_1,\ldots,k_d,n$ are non-negative integers with d being an arbitrary positive integer:

$$(B_{k_1}+\cdots+B_{k_d})^n=\sum_{\substack{l_1+\cdots+l_d=n\\ l_1,\ldots,l_d\geq 0}}\binom{n}{l_1,\ldots,l_d}B_{k_1+l_1}\ldots B_{k_d+l_d}, \tag{1.6}$$

proved an existence theorem for Euler-type formulas, and showed some explicit expressions for $d=3$ in (1.6). Recently, Bayad and Komatsu [6] studied the higher order zeta function and discovered its values at non-positive integers can be determined by

$$(B_{k_1}(x)+\cdots+B_{k_d}(x))^n=\sum_{\substack{l_1+\cdots+l_d=n\\ l_1,\ldots,l_d\geq 0}}\binom{n}{l_1,\ldots,l_d}B_{k_1+l_1}(x)\ldots B_{k_d+l_d}(x), \tag{1.7}$$

where $k_1,\ldots,k_d,n$ are non-negative integers with d being a positive integer. In particular, Bayad and Komatsu [6] showed that for positive integer d and non-negative integers $k_1,\ldots,k_d,m,n$,

$$\begin{aligned}&\sum_{\substack{k_1+\cdots+k_d=m\\ k_1,\ldots,k_d\geq 0}}\binom{m}{k_1,\ldots,k_d}(B_{k_1}(x)+\cdots+B_{k_d}(x))^n\\ &=\sum_{k=0}^{m}\binom{m}{k}(-1)^{k+d}x^{m-k}\frac{(n+k)!}{(n+k+d)!}B_{m+n+d}^{(d)}(x),\end{aligned} \tag{1.8}$$

where the higher order Bernoulli polynomials satisfy the following recurrence relation (see, e.g., [5,6]):

$$B_{n+d}^{(d)}(x)=(n+d)\binom{n+d-1}{d-1}\sum_{i=0}^{d-1}(-1)^i\binom{n-1}{i}B_{d-i-1}^{(d)}(x)\frac{B_{n+i+1}(x)}{n+i+1}. \tag{1.9}$$

The above formulas (1.8) and (1.9) are very interesting and recover the famous Euler's formula on the classical Bernoulli numbers and Dilcher's results stated in [10] on the classical Bernoulli polynomials.

Motivated and inspired by the work of Bayad and Komatsu [6], in this paper we establish a higher-order convolution for the Bernoulli polynomials of the second kind instead of the classical Bernoulli polynomials in the left hand side of (1.8) by making use of the generating function methods and summation transform techniques.

This paper is organized as follows. In the second section, we give the higher-order convolution for the Bernoulli polynomials of the second kind, and present some illustrative special cases as well as immediate consequences of the main result. The third section is contributed to the proof of the main result by applying the generating function methods and summation transform techniques.

2. The statement of results

We now state our main result, as follows.

Theorem 2.1. *Let d be a positive integer and let $y=x_1+\cdots+x_d$. Then, for non-negative integers m, n,*

$$\begin{aligned}&\sum_{\substack{k_1+\cdots+k_d=m\\ k_1,\ldots,k_d\geq 0}}\binom{m}{k_1,\ldots,k_d}(b_{k_1}(x_1)+\cdots+b_{k_d}(x_d))^n\\ &=\frac{(-1)^{d-1}(m+n)!}{(d-1)!}\sum_{i=0}^{d-1}S(d-1,i)\sum_{j=0}^{m+n-d}\binom{y+i}{m+n-d-j}\frac{b_{i+j+1}(y)}{(i+j+1)\cdot j!},\end{aligned} \tag{2.1}$$