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Generalized Szász–Mirakyan operators involving Brenke type polynomials

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ABSTRACT

The aim of the present paper is to introduce generalized Szász–Mirakyan operators including Brenke type polynomials and investigate their approximation properties. We obtain convergence properties of our operators with the help of Korovkin's theorem and the order of convergence by using a classical approach, the second modulus of continuity and Peetre's K-functional. We also give asymptotic formula and the convergence of the derivatives for these operators. Furthermore, an example of Szász–Mirakyan operators including Gould–Hopper polynomials is presented. In the end, we show graphical representation.

1. Introduction

The classical Szász–Mirakyan operators are defined as [11]

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $n \in \mathbb{N},\ \ x \geq 0$, and $f \in C[0,\infty)$ have an important role in the approximation theory, and their approximation properties have been investigated by many researchers. Brenke type polynomials [3] have generating functions of the form

$$C(u)D(xu) = \sum_{k=0}^{\infty} p_k(x)u^k \tag{1.2}$$

where C and D are analytic functions:

$$C(u) = \sum_{s=0}^{\infty} c_s u^s, \quad c_0 \neq 0, \tag{1.3}$$

$$D(u) = \sum_{s=0}^{\infty} d_s u^s, \quad d_s \neq 0 (s \geq 0), \tag{1.4}$$

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and have the following explicit relation:

$$p_k(x) = \sum_{s=0}^{k} c_{k-s} d_s x^s, \quad k = 0, 1, 2, \ldots, \tag{1.5}$$

Using the following restrictions:

1. $C(1) \neq 0,\ \frac{c_{k-s} d_s}{C(1)} \geq 0, \quad 0 \leq s \leq k,\ k = 0, 1, 2, \ldots$
2. D: $[0, \infty) \to (0, \infty)$,
3. (1.1) and the power series (1.2) and (1.3) converge for $|u| < R\,(R > 1)$.

Varma et al. [20] introduced the following positive linear operators involving the Brenke type polynomial

$$L_n(f; x) := \frac{1}{C(1)D(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.6}$$

where $n \in \mathbb{N}$ and $x \geq 0$.

Let $D(u) = e^u$ and $C(u) = 1$. The operators (1.6) reduce to the operators given by (1.1). Recently, Mursaleen and Ansari [17] introduced Chlodowsky variant of Szász operators with the help of Brenke type polynomials as follows:

$$L_n^*(f; x) := \frac{1}{C(1)D(\frac{nx}{d_n})} \sum_{k=0}^{\infty} p_k\left(\frac{nx}{d_n}\right) f\left(\frac{kd_n}{n}\right), \tag{1.7}$$

where (d_n) is a positive increasing sequence such that

$$\lim_{n\to\infty} d_n = \infty, \quad \lim_{n\to\infty} \frac{d_n}{n} = 0.$$

Walczak, in [21] introduced a generalization of (1.1) given as follows:

$$S_n[f; c_n, d_n, q, x] := \sum_{k=0}^{\infty} s_k(c_n x) f\left(\frac{k}{d_n + q}\right), \tag{1.8}$$

where $s_k(c_n x) = e^{-c_n x} \frac{(c_n x)^k}{k!}$, $q \geq 0$ is a fixed number, $(c_n)_1^{\infty}$ and $(d_n)_1^{\infty}$ are given increasing and unbounded numerical sequences such that $d_n \geq c_n \geq 1$, and $(c_n/d_n)_1^{\infty}$ is non-decreasing and

$$\frac{c_n}{d_n} = 1 + o\left(\frac{1}{d_n}\right).$$

In this paper, we define the following generalized Szász–Mirakyan operators including Brenke type polynomials in the form

$$S_n^*[f; c_n, d_n, q, x] := \frac{1}{C(1)D(c_n x)} \sum_{k=0}^{\infty} p_k(c_n x) f\left(\frac{k}{d_n + q}\right), \tag{1.9}$$

1. **Case 1.** Let $c_n = d_n = n$, $q = 0$, $C(u) = 1$ and $D(u) = e^u$, then (1.9) reduces to (1.1).
2. **Case 2.** Let $c_n = d_n = n$ and $q = 0$, then (1.9) reduces to (1.6).
3. **Case 3.** Let $C(u) = 1$ and $D(u) = e^u$, then (1.9) reduces to (1.8).

Recently, Taşdelen et al. [18] and Aktaş et al. [1] have discussed different linear positive operator using Brenke Type Polynomials and Gairola et al. [5] have studied rate of approximation by finite iterates of q-Durrmeyer operators Mishra et al. [15] studied on inverse result in simultaneous approximation by Baskakov–Durrmeyer–Stancu operators. Very recently various generalization of Szász–Mirakjan operators have been studied by Mishra et al. [14], Gandhi et al. [7] and Mishra and Gandhi [13] and Mishra et al. [16] and Gairola et al. [6] have discussed approximation properties of linear positive operators.

The purpose of this paper is to present generalized Szász–Mirakyan operators including Brenke type polynomials and investigate their approximation properties. We obtain convergence properties of our operators with the help of Korovkin's theorem and the order of convergence by using a classical approach, the second modulus of continuity and Peetre's K-functional. We also give asymptotic formula and the convergence of the derivatives for these operators. Furthermore, an example of Szász–Mirakyan operators including Gould–Hopper polynomials is presented. Also, we take

$$\lim_{z\to\infty} \frac{D^{(k)}(z)}{D(z)} = 1, \quad \text{for} \quad k \in \{1, 2, 3, \ldots s\}. \tag{1.10}$$

2. Approximation properties of operators 1.9

Using equality (1.2) and the fundamental properties of the S_n^* operators, we get the following lemmas which is helpful to obtain the main theorem.