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Nonstandard finite differences for a truncated Bratu–Picard model

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ABSTRACT

In this paper, we consider theoretical and numerical properties of a nonlinear boundary-value problem which is strongly related to the well-known Gelfand–Bratu model with parameter λ. When approximating the nonlinear term in the model via a Taylor expansion, we are able to find new types of solutions and multiplicities, depending on the final index N in the expansion. The number of solutions may vary from 0, 1, 2 to ∞. In the latter case of infinitely many solutions, we find both periodic and semi-periodic solutions. Numerical experiments using a non-standard finite-difference (NSFD) approximation illustrate all these aspects. We also show the difference in accuracy for different denominator functions in NSFD when applied to this model. A full classification is given of all possible cases depending on the parameters N and λ.

1. Introduction

In this paper we consider smooth (continuous) solutions of the following truncated Bratu–Picard (tBP) model:

$$\begin{cases} u''(x) + \lambda \sum_{n=0}^{N} \frac{[u(x)]^n}{n!} = 0, \quad x \in [0,1], \quad \lambda \in \mathbb{R}, \quad N \in \mathbb{N} \cup \{0\} \cup \{\infty\}, \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

For $N = \infty$ and $\lambda \geq 0$, this yields the classical Gelfand–Bratu model [5,6,11,23] for which an exact solution is known, see among others [7] and Section 2.4. Traditionally, the names Bratu [5,6] and, sometimes, Gelfand [11] are coupled to this model. However, we propose to use the name Picard as well, since we found by performing a historical literature study that he was the first one who actually introduced the model, with non-unique solutions. His report on this model appeared two decades earlier than Bratu [5]. For more information on this observation, we would like to refer to the four-page note by Picard in [23].

The classical Gelfand–Bratu (GB) problem ($N = \infty$ and $\lambda \geq 0$) is a nonlinear elliptic (partial) differential equation, which finds, for example, its applications in combustion theory (the thermal ignition of a chemically active mixture of gasses) [4]. Other application areas of the GB problem appear in elasticity theory (membrane buckling) [26], in astronomy (gravitational equilibrium of polytropic stars and the Chandrasekhar model of the expansion of the universe) [8], in thermo-electro-hydrodynamic models and in nanotechnology (electrospinning processes) [28].

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In previous numerical studies of the classical Gelfand–Bratu problem, several numerical methods have been proposed and compared to the exact solution. Most of these methods converge only to one of the two solutions of the model. Different computational techniques were compared with each other. For example, finite difference methods and multigrid methods are used in [21]. A direct shooting method and a Lie-group shooting method were presented in [1,7]. Further, perturbation iterations, parameter perturbations and parameter splines were implemented in [15]. Phase plane solutions for perturbation problems were given in [25]. Boyd's method [14] succeeded in giving both the lower and upper solutions, in case of multiple solutions. Buckmire [7] applied Mickens nonstandard finite difference method (NSFD) and compared the performances of the Adomian decomposition method, Boyd's pseudospectral method, nonlinear shooting method and standard finite difference (SFD) and NSFD methods. Buckmire reported that the NSFD method may converge to both solutions (the lower and the upper one) and is more accurate than SFD. A smart NSFD scheme for second order nonlinear boundary value problem has been discussed in Erdogan [10]. The related, more general, compact exponentially fitted method is used in [22] and SFD and NSFD approaches are considered as special cases. Recently an iterative finite difference method for solving the GB-model has been discussed in [13]. Mohsen [20] presented a straightforward solution technique of the 1D planar Bratu problem with different treatments of the resulting nonlinear system of equations by using SFD and NSFD methods. Mohsen recommended a simple sinusoidal function as an initial guess for NSFD which provides more accurate results.

Motivated by the recent articles of Mohsen [20] and Buckmire [7], we present a new and extended study of the tBP model of which the GB-model is a special case. We consider several cases of model (1.1) and investigate properties of the solution $u_\lambda^N(x)$ which depend on the parameter λ and the index N. We present several theoretical properties of the solution in which each solution of the model (1.1) has exactly one maximum and is symmetric at $x = \frac{1}{2}$ for $\lambda > 0$. Previous articles considered only positive solutions but we consider all smooth solutions, where some of them are periodic and others are semi periodic. We also show theoretically and numerically that a unique solution exists for $\lambda \leq 0$. We work out asymptotic expressions, to show the behavior of the solution for small and large values of the parameter λ. We present a further study of the NSFD scheme for iteratively solving the resulting nonlinear systems by choosing a simple sinusoidal function having the appropriate amplitude, as an initial guess. We observe that NSFD has a similar simplicity as an SFD approximation but it is slightly more accurate, in most cases. Numerical experiments show that a large number of solutions can be obtained which are either periodic or semi-periodic. In fact, the theory shows that infinitely many may exist. This is explored in an upcoming paper [30]. Graphically we also present the bifurcation nature of all possible cases of the tBP model (1.1). In literature, mainly positive solutions are considered for the $\lambda \geq 0$-case. In this paper, we give a full classification of all solution types of model (1.1), both positive or negative and (semi)-periodic, for all $\lambda \in \mathbb{R}$.

The paper is organized as follows. In Section 2, we present several theoretical, analytical and asymptotic properties of the solution and we also discuss exact solutions for some special cases of the truncated Bratu–Picard model (1.1). The SFD and NSFD approximations are worked out in Section 3. In Section 4, numerical experiments are performed to discuss the numerical aspects of the different cases of the truncated Bratu–Picard problem. All types of possible shapes of solutions (periodic and semi-periodic) and bifurcations are displayed in Section 5. In Section 6, we summarize the theoretical and numerical results.

2. Properties of the solution

We will denote solutions of the model (1.1) simply by $u(x)$ or, when it is appropriate, by $u_\lambda^N(x)$ to stress their dependence on the parameters λ and N.

2.1. General properties

We distinguish between several cases. For this, we define the following two subsets of $\mathbb{N}$:

$\mathcal{N}_2 := \{2, 4, 6, 8, \ldots\}$,

$\mathcal{N}_3 := \{3, 5, 7, 9, 11, \ldots\}$.

Note, that the special cases $N = 0$, $N = 1$ and $N = \infty$ will be treated separately. Further, it is useful to define the functions

$$f(u) = f_N(u) := \sum_{n=0}^{N} \frac{u^n}{n!} \quad \text{and} \quad F(u) := \int_0^u f(\omega)\, d\omega.$$

In the following part, we describe and prove a series of analytical and asymptotic properties of the solution $u(x)$ of model (1.1).

Lemma 1. *For $N \in \mathcal{N}_2$, we have $f_N(u) > 0$ for all $u \in \mathbb{R}$, whereas for $N \in \mathcal{N}_3$, $f_N(u) > 0$ for all positive u and a unique value $\tilde{u} < 0$ exists with $f_N(\tilde{u}) = 0$ (see also the four graphs in Fig. 1).*

Proof. Note that, for $u \geq 0$, automatically we find $f_N(u) > 0$, since all single terms are positive and $f_N(0) = 1$. Next, assume that $u < 0$. Then $f'_N(u) = f_{N-1}(u)$ and $f_N(u) = f_{N-1}(u) + \frac{u^N}{N!}$. For $N \in \mathcal{N}_2$, we observe that $\lim_{u \to -\infty} f_N(u) = +\infty$. Suppose that there exists a value $\tilde{u} < 0$ such that $f_N(\tilde{u}) \leq 0$. Then, there must be a $u^* < 0$ such that $f_N(u^*) \leq 0$ and $f'_N(u^*) = 0$. At this