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Stability analysis of a parametric family of seventh-order iterative methods for solving nonlinear systems☆

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ABSTRACT

In this paper, a parametric family of seventh-order of iterative method to solve systems of nonlinear equations is presented. Its local convergence is studied and quadratic polynomials are used to investigate its dynamical behavior. The study of the fixed and critical points of the rational function associated to this class allows us to obtain regions of the complex plane where the method is stable. By depicting parameter planes and dynamical planes we obtain complementary information of the analytical results. These results are used to solve some nonlinear problems.

1. Introduction

Nonlinear systems arise in different areas of scientific computing and engineering computations, many problems modeled in science and applied problems are in the form of equations or systems of nonlinear equations. So, the analysis of these type of equations is an interesting field of study. Let us consider the system of nonlinear equations $F(x)=0$, where $F: D\subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$. As nonlinear systems are difficult to solve, the solution $\bar{x}$ is usually approximated by a fixed point function $G: D\subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defining a fixed point iteration scheme. There are many root-finding iterative schemes to solve systems of nonlinear equations. The famous second order Newton's method, is a powerful iterative scheme for solving nonlinear equations and systems. But, in recent years, some researchers have proposed new iterative methods with higher order and better efficiency as an alternative to classical Newton's scheme.

Sharma and Arora in [17] proposed an eighth-order method that is a three step scheme, denoted by NM_8, as follows

$$\begin{aligned} y^{(k)} &= x^{(k)} - \left[F'(x^{(k)})\right]^{-1}F(x^{(k)}),\\ z^{(k)} &= y^{(k)} - \left[\frac{13}{4}I - G(x^{(k)})\left(\frac{7}{2}I - \frac{5}{4}G(x^{(k)})\right)\right]\left[F'(x^{(k)})\right]^{-1}F(y^{(k)}),\\ x^{(k+1)} &= z^{(k)} - \left[\frac{7}{2}I - G(x^{(k)})\left(4I - \frac{3}{2}G(x^{(k)})\right)\right]\left[F'(x^{(k)})\right]^{-1}F(z^{(k)}) \end{aligned} \quad (1)$$

where $G(x^{(k)}) = \left[F'(x^{(k)})\right]^{-1}F'(y^{(k)})$ and I is the identity matrix.

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In [7], Cordero et al. designed two eighth-order methods that the first one is describe as

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - \left[F'(x^{(k)})\right]^{-1}F(x^{(k)}),\\
z^{(k)} &= y^{(k)} - \left[\frac{5}{4}I - \frac{1}{2}\left[F'(y^{(k)})\right]^{-1}F'(x^{(k)}) + \frac{1}{4}\left(\left[F'(y^{(k)})\right]^{-1}F'(x^{(k)})\right)^2\right]\left[F'(y^{(k)})\right]^{-1}F(y^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left[\frac{3}{2}I - \left[F'(y^{(k)})\right]^{-1}F'(x^{(k)}) + \frac{1}{2}\left(\left[F'(y^{(k)})\right]^{-1}F'(x^{(k)})\right)^2\right]\left[F'(y^{(k)})\right]^{-1}F(z^{(k)}).
\end{aligned} \tag{2}
$$

and is denoted by CCGT1. The second one, denoted by CCGT2, has as iterative expression

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - \left[F'(x^{(k)})\right]^{-1}F(x^{(k)}),\\
z^{(k)} &= y^{(k)} - \left[\frac{1}{4}I + \frac{1}{2}\left[F'(y^{(k)})\right]^{-1}F'(x^{(k)}) + \frac{1}{4}\left(\left[F'(y^{(k)})\right]^{-1}F'(x^{(k)})\right)^2\right]\left[F'(x^{(k)})\right]^{-1}F(y^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left[\frac{1}{2}I + \frac{1}{2}\left(\left[F'(y^{(k)})\right]^{-1}F'(x^{(k)})\right)^2\right]\left[F'(x^{(k)})\right]^{-1}F(z^{(k)}).
\end{aligned} \tag{3}
$$

According to Ostrowski's efficiency index, defined as $I = p^{1/d}$, (see [16]) where p is the order of convergence and d is the number of function evaluations per iteration, all these methods show better efficiency than Newton's scheme.

But, increasing order of convergence usually makes decreasing the radius of convergence, and since determining the regions where the initial guess shows better convergence behavior is important, so the study of dynamical behavior of the iterative methods is very helpful. Dynamical analysis of iterative methods for nonlinear systems is so complicated (see [6,9]) and most of the times is impossible, but the study of the behavior of the iterative method in scalar case and in the complex plane is an interesting field of study. Moreover even in scalar case for higher order methods studying dynamical behavior is a difficult task and sometimes is impossible due to the high degree of the rational functions involved. The advent of computers in last decades, made it practically possible to study the structure of the dynamical and parameter planes of iterative methods closely in some special cases, since large amount of computational power is needed to obtain their precise shape, that can be easily performed in computers.

The main aim of this analysis is finding the regions in the complex plane where our function shows better performance when converges to the zeros of the function. But even in the scalar case, finding stable regions for a high order iterative method is not easy. High order iterative methods, even for simple scalar nonlinear function $f(z)$, usually results in a high degree fixed point operator, since the key of stability analysis is the study of the fixed point operator.

In this paper, we propose the following family of seventh-order iterative methods to solve systems of nonlinear equations, whose iterative expression is

$$
\begin{aligned}
y^{(k)} &= x^{(k)} - \left[F'(x^{(k)})\right]^{-1}F(x^{(k)}),\\
z^{(k)} &= y^{(k)} - \frac{1}{\beta}\left[F'(x^{(k)})\right]^{-1}F(y^{(k)}),\\
w^{(k)} &= z^{(k)} - \left[F'(x^{(k)})\right]^{-1}\left((2-1/\beta-\beta)F(y^{(k)}) + \beta F(z^k)\right),\\
x^{(k+1)} &= w^{(k)} - G(t^{(k)})\left[F'(x^{(k)})\right]^{-1}F(w^{(k)}),
\end{aligned} \tag{4}
$$

where $t^{(k)} = I - \frac{1}{\beta}\left[F'(x^{(k)})\right]^{-1}[y^{(k)}, z^{(k)}; F]$ and $G : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ is a matrix weight function that is chosen in order to obtain the seventh-order of convergence. Also $[.,.;F] : \Omega\times\Omega \subset \mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n)$ is the divided difference operator of F on $\mathbb{R}^n$, defined as (see [15])

$$[x, y; F](x-y) = F(x) - F(y), \quad \text{for any } x, y \in \Omega.$$

Let us remark that the first three steps correspond to a fourth-order parametric family whose convergence and stability was analyzed in [8]. In this paper, we are going to analyze the dynamical behavior of class (4) on scalar functions. This analysis will be made on quadratic polynomial, as they are the simplest nonlinear functions, and it will give us important information about the stability of the family in terms of the value of the parameter and its dependence on the initial estimations used.

The rational function associated with a subclass of (4) on the quadratic polynomial $p(z) = (z-a)(z-b)$ is used in the following and denoted by $O_p(z)$. The obtained results can be extrapolated, up to some extent, to more complicated nonlinear function, as can be observed in related research [1,5,11–14] and in our numerical tests.

Now, we recall some dynamical concepts that we use in this paper (see [2]). Let $R : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational function, where $\widehat{\mathbb{C}}$ is the Riemann sphere, the orbit of a point $z_0 \in \widehat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}. \tag{5}$$

where R^k denotes the kth composition of the map R with itself. We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of the orbits. The point z_0 is called a fixed point if $R(z_0) = z_0$ and is a periodic