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Matching of 5-γ-critical leafless graph with a cut edge[☆]

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ABSTRACT

Given a graph $G=(V,E)$, a subset S of V is a dominating set of G if every vertex in $V\backslash S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. A graph G is said to be k-γ-critical if $\gamma(G)=k$, but $\gamma(G+e)<k$ for each edge $e\in E(\overline{G})$, where $\overline{G}$ is the complement of G. In this paper, we first provide the structure of k-γ-critical connected graphs with a nontrivial cut edge. Then we establish that each 5-γ-critical leafless connected graph of even order contains a perfect matching.

1. Introduction

In this paper, all graphs will be finite, simple and connected. Let $G=(V,E)$ be a graph. The complement of G is denoted by $\overline{G}$. For a subset $S\subseteq V$, $G[S]$ is a subgraph of G induced by S. For a vertex v in G, the set $N(v)=\{u\in V\mid uv\in E\}$ is called the *open neighborhood* of v and $N[v]=N(v)\cup\{v\}$ is the *closed neighborhood* of v. For a subset $S\subseteq V$, $N(S)=\bigcup_{v\in S}N(v)$ is the *open neighborhood* of S and $N[S]=N(S)\cup S$ is the *closed neighborhood* of S. Let H be a subgraph of G and $v\in V(G)$. We will use $H+v$ and $H-v$ to denote the subgraph $G[V(H)\cup\{v\}]$ and $G[V(H)\setminus\{v\}]$ of G, respectively. Similarly, the graph $G+uv$ arises from G by adding an edge uv between two non-adjacent vertices u and v of G.

A vertex of degree one is called a *leaf*. The edge incident with a leaf is known as a *pendant edge*. A *cut edge* of a connected graph is one whose deletion results in a disconnected graph. A cut edge is said to be a *nontrivial cut edge* if it is not a pendant edge. A graph is called *leafless* if there is no leaves in the graph. Let $G=(V,E)$ be a graph with n vertices. A *matching* of G is a set M of edges such that no two edges are incident with a common vertex. The matching number of G, denoted by $\nu(G)$, is the size of a maximum matching of G. If n is even and $\nu(G)=\frac{n}{2}$, then we will call G has a *perfect matching*. A *near-perfect matching* of G is a set M of edges such that it is incident with all vertices of G except exactly one. Obviously, G has a near-perfect matching if and only if n is odd and $\nu(G)=\frac{n-1}{2}$. If $G-v$ has a perfect matching for every choice of $v\in V(G)$, then G is said to be *factor-critical*. If G is factor-critical, then G has a near-perfect matching. A set of vertices of G is called an *independent set* if no two of its elements are adjacent.

A *semicomplete digraph* is a biorientation of a complete graph and a *tournament* is an orientation of a complete graph. Let $D=(V,E)$ be a digraph of order n. A path P in D is an alternating sequence $v_1v_2\cdots v_k$ such that $v_iv_{i+1}\in E(D)$ for $1\le i\le k-1$ and the vertices of P are distinct. P is called a *Hamilton path* of D if $k=n$. It is well known if D is a tournament, then D contains a Hamilton path.

A subset S of V is a *dominating set* of G if $N[S]=V$. For S, $T\subseteq V(G)$, we say that S dominates T, denoted by $S\succ T$, if $T\subset N_G[S]$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. G is said to be

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k-γ-*critical* if $\gamma(G)=k$, but $\gamma(G+e)<k$ for each edge $e\in E(\overline{G})$. It is easy to find that a graph is 1-γ-critical if and only if it is complete.

This concept of k-γ-critical graph was first introduced in 1983 by Sumner and Blitch [6]. In [6], 2-γ-critical graphs and 3-γ-critical disconnected graphs were characterized. Since then k-γ-critical graphs have been paid a lot of attention (see for example [1–6,8]).

The problem of prefect matching in k-γ-critical graph is delicate and interesting. There are many results about matching for $k=3$, see [6] and [2]. For example, it was proved that every connected 3-γ-critical graph with even order has a perfect matching by Sumner and Blitch [6]. In 2010, Ananchuen et al. [3] considered the k-γ-critical graphs for $k\geq 4$. They first gave a characterization of k-γ-critical connected graphs containing at least $k-1$ leaves and then showed that if G is a k-γ-critical connected graph of even order containing at least $k-2$ leaves for $k\geq 4$, then G has a perfect matching. They also showed that if G is a 4-γ-critical graph of even order with a cut vertex, then G has a perfect matching. These results partially resolved a problem posed by Sumner and Wojcicka in [7].

In this paper, we will concentrate on 5-γ-critical graph with a nontrivial cut edge. We will provide the structure of a k-γ-critical graphs with a nontrivial cut edge. Then we show that a 5-γ-critical leafless graph of even order with a cut edge has a perfect matching if it is connected. Our result also partially resolves the problem posed by Sumner and Wojcicka in [7] in which they asked whether every k-γ-critical graph of even order contains a perfect matching for $k\geq 4$.

2. Preliminaries

In this section, we first cite some results which will be used to prove our theorems. For a pair of non-adjacent vertices u and v in G, we denote a minimum dominating set in $G+uv$ by D_{uv} for short. Note that the minimum dominating set in $G+uv$ is not unique.

Lemma 1 [3]**.** *Let u and v be a pair of non-adjacent vertices in a k-γ-critical graph G. Then*

(1) $|D_{uv}|=k-1$ *and* $|D_{uv}\cap\{u,v\}|=1$*;*
(2) *If* $D_{uv}\cap\{u,v\}=\{u\}$, *then* $D_{uv}\cap N_G[v]=\emptyset$.

Theorem 2 [6]**.** *A graph G is 2-γ-critical if and only if $\overline{G}=\cup_{i=1}^{n}K_{1,s_i}$, where $n\geq 1$, $s_i\geq 1$.*

Theorem 3 [5]**.** *If a is a cut vertex of a k-γ-critical graph G, then $G-a$ has exactly two components.*

Theorem 4 [5]**.** *Let G be a k-γ-critical graph with a cut vertex a and C_1 and C_2 be the components of $G-a$. Further let $A=G[V(C_1)\cup\{a\}]$ and $B=G[V(C_2)\cup\{a\}]$. Then $\gamma(A)+\gamma(B)=k$.*

A vertex v of a graph G is said to be *critical* if $\gamma(G-v)<\gamma(G)$ and it is *stable* if $\gamma(G-v)=\gamma(G)$.

Theorem 5 [5]**.** *Let G be a k-γ-critical graph. Then*

(1) *Every vertex of G is either critical or stable.*
(2) *If a is a cut vertex, then a is stable.*
(3) *If C is the set of all stable vertices of G, then $G[C]$ is complete.*

Theorem 6 (Tutte)**.** *A connected graph of even order has a perfect matching if and only if G does not contain a subset $S\subset V(G)$ such that $G-S$ has at least $|S|+2$ odd components.*

3. k-γ-critical connected graphs with a nontrivial cut edge

Throughout this section, we suppose G is a k-γ-critical connected graph with a nontrivial cut edge $e=ab$, where $k\geq 2$. We will give a characterization of such graphs.

Since $e=ab$ is a nontrivial cut edge, there are $c\in N(a)\backslash\{b\}$ and $d\in N(b)\backslash\{a\}$. Let A and B be the components of $G-e$ with $a\in V(A)$ and $b\in V(B)$, respectively. Then $c\in V(A)$ and $d\in V(B)$ (see Fig. 1). We have the following results.

Theorem 7.

(1) *If $\gamma(A)=1$, then $A=K_2$.*
(2) *If $\gamma(A)=2$, then $A-a$ is 2-γ-critical. Furthermore, $\overline{A-a}=\cup K_2$.*

Proof. Since ab is a cut edge, we have that a, b are stable vertices of G and v is critical for each $v\in V(G)\setminus\{a,b\}$ by Theorem 5.

Claim 1. *$\gamma(G)=\gamma(A)+\gamma(B)$, $\gamma(A)=\gamma(A-a)=\gamma(A+b)$ and $\gamma(B)=\gamma(B-b)=\gamma(B+a)$.*

Proof of Claim 1. Let D_A and D_B be the minimum dominating set of A and B, respectively. Then $D_A\cup D_B$ is a dominating set of G. So we have $\gamma(A)+\gamma(B)\geq\gamma(G)$. □

Now we consider the minimum dominating set D_{ad} in $G+ad$. By Lemma 1, $D_{ad}\cup\{a, d\}$ is a minimum dominating set of G. Also $(D_{ad}\cup\{a, d\})\cap A\succ A$ and $(D_{ad}\cup\{a, d\})\cap B\succ B$. So we have $\gamma(A)+\gamma(B)\leq\gamma(G)$. Thus $\gamma(G)=\gamma(A)+\gamma(B)$.