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Some geometrical properties of free boundaries in the Hele-Shaw flows

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ABSTRACT

In this paper, we are concerned with certain geometric properties of the moving boundary in the case of two-dimensional viscous fluid flows in Hele-Shaw cells under injection. We study the invariance in time of free boundary for such a bounded flow domain under the assumption of zero surface tension. By applying various results in the theory of univalent functions, we consider the invariance in time of starlikeness of a complex order, almost starlikeness of order $\alpha \in [0, 1)$, and almost spirallikeness of type $\gamma \in (-\pi/2, \pi/2)$ and order $\alpha \in (0, \cos\gamma)$. This work complements recent work on planar Hele-Shaw flow problems in the case of zero surface tension.

1. Introduction

Hele-Shaw cell is an investigation instrument for the study of the flow of a viscous incompressible fluid between two flat plates separated by a very small distance. Initially, the fluid occupies a phase domain with free boundary, and through a given point more fluid is injected or removed. Then the fluid domain changes its shape in time. The Hele-Shaw flow problem consists of determining the evolution in time of the shape of the domain, as well as the corresponding fluid flow. A very good overview of this problem may be found in the excellent monographs [15,16] (see also [1,36]).

The invariance in time of some geometric properties of the moving boundary of a two-dimensional flow of a viscous fluid in Hele-Shaw cells under injection was studied by a number of authors (see [4–7,10,17,22,34,35,37]).

In the case of bounded domains and in the absence of surface tension, the first related results in were obtained by Prokhorov, Vasil'ev and Hohlov in [17]. They applied methods of geometric function theory to study the inner problem in a bounded simply connected domain, and proved that if the initial domain has a starlike boundary with respect to the origin, then the moving boundary has the same property during the existence time of the solution. In the same paper [17], the authors proved that the geometric notions of convexity and close-to-convexity are not preserved in time, in the case of the inner problem. More recently, the above work was extended to other families of univalent functions. Significant results in this direction were obtained by Vasil'ev [34] who proved the invariance in time of strongly starlikeness of order α, and convexity in the direction of the real axis, in the case of the inner problem and assuming the surface tension is zero. Gustafsson, Prokhorov and Vasil'ev [14] proved that in the absence of surface tension, if the initial domain is bounded and strongly starlike of order $\alpha \in (0, 1]$, then one obtains a subordination chains of strongly starlike of order $\alpha(t)$ domains,

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where $\alpha(\cdot)$ is a decreasing function (see also [23]). As a consequence, the authors deduced in [14] that the infinite life time in the case of classical starlike Hele-Shaw flow problem is infinite. Curt and Fericean [5] proved the invariance in time of the notion of Φ-likeness, while Curt, Fericean and Groşan [6] deduced that the notion of strongly Φ-likeness of order α is also a geometric invariant during the existence time of solution, in the case of bounded domains with analytic and smooth boundaries. Recently, Curt (see [2,4]) obtained other invariant geometric properties in Hele-Shaw cells, such as starlikeness of a given order, and α-convexity, $\alpha \geq 0$ (in the case of bounded domains). A partial solution to an open problem of Vasil'ev [34], concerning the preservation in time of the notion of starlikeness of order α, was obtained in [4].

The first results related to the preservation in time of geometric properties in the case of unbounded domains (with bounded complement) were obtained Entov and Etingov [7]. They proved that if the domain at the initial moment has a convex complement, then the family of domains occupied by the fluid at different moments of time has the same property as long as the solution of the Hele-Shaw problem exists. Assuming the absence of surface tension, their study was extended to other families of univalent functions (see [2,5,15,30]).

We mention that various results concerning Hele-Shaw flows in the case of small non-zero surface tension may be found in [3,5,6,10,15,32,34,35,37]. A layer potential approach of a Hele-Shaw flow problem in higher dimensions has been recently considered by Kohr and Pintea [20]. They treated a boundary value problem that describes the evolution of a fluid flow domain due to the injection with fluid in a viscous incompressible fluid in the presence of a solid obstacle Other applications of layer potential theory to elliptic boundary value problems have been recently obtained by Kohr et al. (see [18,19]; see also [21]). Higher dimensional approaches of Hele-Shaw models were considered in [8,9,20] (see also [16]). We also mention that significant applications concerning Hele-Shaw flow problems in various domains have been obtained in [31,33].

In this paper, we consider the flow of a viscous fluid in the $\mathbb{C}$-plane Hele-Shaw cell under injection through a source with constant strength $Q<0$, which is located at the origin. Let $\Omega(t)$ be the bounded simply connected domain in $\mathbb{C}$, which contains the origin and is occupied by the fluid at the time $t \geq 0$. Assume that the boundary $\partial\Omega(t)$ of $\Omega(t)$ is an analytic and smooth curve, for all $t \geq 0$. In view of the Riemann mapping Theorem, there exists a unique univalent mapping $f(\cdot, t)$ of the unit disk $\mathbb{U} = \{z : |z| < 1\}$ onto $\Omega(t)$ such that $f(0,t) = 0$ and $f'(0, t) > 0$. Also, the univalent mapping $f(\cdot, t)$ provides the parametrization of the free boundary $\partial\Omega(t)$,

$$\partial\Omega(t) = \left\{ f(e^{i\theta}, t) : \theta \in [0, 2\pi) \right\}, \quad t \geq 0.$$

Throughout the paper, we shall use the following notations: $f'(z,t) = \frac{\partial f}{\partial z}(z,t)$ and $\dot{f}(z,t) = \frac{\partial f}{\partial t}(z,t)$.

Assuming the absence of surface tension, the free boundary $\partial\Omega(t)$ satisfies the Polubarinova-Galin equation (see [11,27,28]):

$$\mathrm{Re}\left[\dot{f}(z,t)\overline{zf'(z,t)}\right] = -\frac{Q}{2\pi}, \quad |z| = 1. \tag{1.1}$$

We remark that the above equation is related to the Loewner-Kufarev equation (see e.g., [15,34]). It is well known that the Loewner-Kufarev PDE has multiple applications in the theory of univalent functions on the unit disk in $\mathbb{C}$ (see e.g., [29]).

Recall that a classical solution of (1.1) on the interval $[0, T)$ is a mapping $f(z, t)$ such that $f(\cdot, t)$ is analytic and univalent on a neighborhood of $\overline{\mathbb{U}}$ for $t \in [0, T)$, and $f(z, \cdot)$ is of class C^1 on $[0, T)$ for $z \in \overline{\mathbb{U}}$. In this case, the number T is called the blow-up time (lifetime) (see, e.g., [15,30]).

Under the assumption that the initial domain $\Omega(0)$ has an analytical and smooth boundary, it is known that the classical solution of the Eq. (1.1) exists and is unique locally in time (see [30,38]; see also [15, 16, Chapter 1]). Various details related to the behavior of the classical solution to the boundary of the unit disk $\mathbb{U}$ may be found in [16,30,38].

The main results of this paper provide the invariance in time of some geometric notions, namely starlikeness of complex order, almost starlikeness of order α, and almost spirallikeness of type γ and order α, in the absence of surface tension.

2. Preliminaries

We are interested in the preservation of geometric properties of flow domains in the Hele-Shaw flow problem. To this end, we study geometric and analytic properties of the corresponding conformal mappings of the unit disc onto the investigated flow domains.

In this section, we recall some notions and results in the theory of univalent functions of one complex variable that are needed in the statements of the main results. Note that the domain D in the following definitions plays the role of the initial fluid domain $\Omega(0)$.

We begin with the following notion of starlikeness of complex order on the unit disk $\mathbb{U}$ (see [26]; see e.g., [12,25]).

Definition 1. Let $f : \mathbb{U} \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f'(0) \neq 0$. Also, let $b \in \mathbb{C} \setminus \{0\}$. We say that f is *starlike of complex order b* if

$$\mathrm{Re}\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] > 0, \quad z \in \mathbb{U}. \tag{2.1}$$

Let $D \subseteq \mathbb{C}$ be a domain that contains the origin. We say that D is starlike of complex order b if there exists a starlike function of complex order b which maps the unit disk $\mathbb{U}$ conformally onto D.