Applied Mathematics and Computation 323 (2018) 106–119

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Modified methods for solving two classes of distributed order linear fractional differential equations

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ARTICLE INFO

Keywords:
Distributed-order linear fractional differential equations
Operational matrix
The Chebyshev polynomials
Picard's method

ABSTRACT

This paper introduces two methods for the numerical solution of distributed order linear fractional differential equations. The first method focuses on initial value problems (IVPs) and based on the αth Caputo fractional definition with the shifted Chebyshev operational matrix of fractional integration. By applying this method, the IVPs are converted into simple linear differential equations which can be easily handled. The other method focuses on boundary value problems (BVPs) based on Picard's method frame. This method is based on iterative formula contains an auxiliary parameter which provides a simple way to control the convergence region of solution series. Several numerical examples are used to illustrate the accuracy of the proposed methods compared to the existing methods. Also, the response of mechanical system described by such equations is studied.

1. Introduction

The distributed order equations are considered as a source of many mathematical physics problems such as [1–12]: viscoelastic model [3], the distributed order oscillator [4], system identification [5,6], diffusion [7,8], complex system [9,10], and many recently engineering applications [11,12]. The existence and the uniqueness of the solution of distributed order equations were discussed in [13–15]. There are few numerical methods were proposed to solve ordinary and partial differential equations of distributed order based on the reproducing kernel method [16], finite and compact difference schemes [17–20], analog equation method [21] and block pulse function [22].

In this paper, two methods are presented for solving two classes of distributed order linear fractional differential equations.

The first method based on the shifted Chebyshev operational matrix of fractional integration to treatment initial value problems in the form

$$\sum_{i=1}^{m}\int_{\alpha_{0,i}}^{\alpha_{n,i}} W_i(\alpha) D_c^{\alpha} u(t)\,d\alpha = f(t),\ \ u^{(k)}(0) = b_k\ \ k = 0, 1, \ldots,\ \alpha_{n,m-1}, \tag{1}$$

where $0 \le \alpha_{0,i} < \alpha_{n,i} < \alpha_{n,m}$ and $D_c^{\alpha} u(t)$ is the Caputo type fractional derivative of order α.

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https://doi.org/10.1016/j.amc.2017.11.047
0096-3003/

The basic ingredients of the proposed method consist of:

1- Replacing the fractional derivative $D_c^\alpha u(t)$ by a single definition of Caputo derivative in any interval $[\alpha_{0,i}, \alpha_{n,i}]$ based on only $\lceil \alpha_{n,m} \rceil$.
2- Converting the problem into linear integer order differential equation with order $\lceil \alpha_{n,m} \rceil$ by use the operational matrix of fractional integration for shifted Chebyshev polynomials.

Also, based on the Picard's method [23–25], a new iterative formula contains an auxiliary parameter for solving two-point boundary value problems is introduced in the form:

$$\int_0^2 W(\alpha) D_c^\alpha u(t) u(t)\, d\alpha = f(t),\ u^{(i)}(0) = b_i,\ u(1) = b,\ i = 0 \text{ or } 1. \tag{2}$$

The basic ingredients of the proposed method consist of:

1- Approximating the non-homogenous term $f(t)$ with a polynomial function using the shifted Chebyshev polynomials.
2- Construct an iterative formula based on Picard method to solve the resulting problem.
3- Discovering and verify region of convergence of a solution series by so called h -curve.

The paper is organized as follows. Section 2 introduces some necessary definitions and fundamentals, and Section 3 discusses the proposed methods as well as how they are used to solve the two classes of distributed order linear fractional differential equations. Section 4 illustrates some numerical results based on the proposed methods with some comparisons with previous methods. Finally, the conclusion messages are summarized in Section 5.

2. Preliminary

Recently, there are a huge motivation and publication records in the research topic of fractional-calculus and its applications in different fields. Although there are many definitions of the fractional order derivative [26], but the most used definition is due to Caputo, which is defined as

$$D_c^\alpha u(t) = J^{r-\alpha} u^{(r)}(t),\ r-1 < \alpha \le r, \tag{3}$$

where $J^p(.)$ is Riemann–Liouville fractional integral operator of order $p \ge 0$ and defined by:

$$J^p u(t) = \begin{cases} \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} u(\tau) d\tau,\ \alpha > 0 \\ u(t),\ \alpha = 0 \end{cases}. \tag{4a}$$

Some of the properties of the operator J^p are:

$$J^p t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+p+1)} t^{p+\gamma},\ J^p J^\gamma (cf(t) + dg(t)) = cJ^{p+\gamma} f(t) + dJ^{p+\gamma} g(t), \tag{4b}$$

where c and d are constants.

Lemma 2.1. [26]. *If $r-1 < \alpha \le r$, $r \in N$ then*

$$J^\alpha D_c^\alpha u(t) = u(t) - \sum_{j=0}^{r-1} u^{(j)}(0) \frac{t^j}{j!}. \tag{5}$$

Recently, Chebyshev polynomials have been integrated with many numerical methods to solve many engineering applications. It is well-known that the Chebyshev polynomials are defined on $[-1, 1]$. Moreover, a kind of Chebyshev polynomials called the shifted Chebyshev polynomials $\mathcal{T}_{L,i}(t)$ over $[0, L]$ have been introduced [27], with three-term recurrence relations as follows:

$$\mathcal{T}_{L,0}(t) = 1,\ \mathcal{T}_{L,1}(t) = \frac{2t}{L} - 1,\ \mathcal{T}_{L,i+1}(t) = 2\left(\frac{2t}{L} - 1\right)\mathcal{T}_{L,i}(t) - \mathcal{T}_{L,i-1}(t),\ i = 1, 2, \ldots \tag{6}$$

The orthogonally condition is

$$\int_0^L \mathcal{T}_{L,i}(t) \mathcal{T}_{L,j}(t) w_L(t) dt = h_j \delta_{ij}, \tag{7}$$

where $w_L(t) = 1/\sqrt{t(L-t)}$ is a weight function and $\delta_{ij}$the Kronecker function, $h_0 = \pi$ and $h_j = \pi/2,\ j \ge 1$. The fractional integral of order p for the shifted Chebyshev vector $\phi_L(t)$ is given by the following Theorem:

Theorem 1. [27]: *Let $\Phi_L(t)$ be the shifted Chebyshev vector and $p > 0$, then*

$$J^p \Phi_L(t) \cong H^{(p)} \Phi_L(t), \tag{8}$$