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An integrable generalization of the D-Kaup–Newell soliton hierarchy and its bi-Hamiltonian reduced hierarchy

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ABSTRACT

We present a new spectral problem, a generalization of the D-Kaup–Newell spectral problem, associated with the Lie algebra $\mathrm{sl}(2,\mathbb{R})$. Zero curvature equations furnish the soliton hierarchy. The trace identity produces the Hamiltonian structure for the hierarchy and shows its Liouville integrability. Lastly, a reduction of the spectral problem is shown to have a different soliton hierarchy with a bi-Hamiltonian structure. The major motivation of this paper is to present spectral problems that generate two soliton hierarchies with infinitely many conservation laws and high-order symmetries.

1. Introduction

The study of soliton equations has been of considerable importance to the understanding of nonlinear phenomenon over the past few decades. In recent years, soliton theory has enriched the understanding of the nature of integrability in partial and ordinary differential equations (see [1]); one way is through the existence of infinitely many conservation laws and symmetries. Constructed from spectral problems associated with matrix Lie algebras, systems of solitons equations often give rise to soliton hierarchies [2–4]. Frequently, these hierarchies possess infinitely many symmetries and conserved functionals. Some hierarchies of this particular type include the Ablowitz-Kaup-Newell-Segur [5], the Kaup-Newell [6], the D-Kaup-Newell [7], the KdV [8], and the Dirac hierarchies [9]. This paper presents two spectral problems that generate different soliton hierarchies; both hierarchies have infinitely many conservation laws and high-order symmetries implying Liouville integrability.

There is a general scheme for soliton hierarchy construction [10,11]. We begin with a matrix spectral problem associated with a matrix loop algebra. Under the assumption of a solution to the stationary zero curvature equation, a series of Lax matrices is introduced. The Lax matrices give rise to the temporal spectral problems. Each temporal spectral problem paired with the original spectral problem form a member in a series of compatibility conditions known as the zero curvature equa-

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tions. After solving the zero curvature equations, we generate the hierarchy of soliton equations. The final step is finding the Hamiltonian structure for the hierarchy using the trace identity [10,11]. This produces a hierarchy of Hamiltonian equations.

We must begin by introducting a matrix loop algebra. The two spectral matrices in this paper are associated with $\mathrm{sl}(2,\mathbb{R})$, a three-dimensional special linear Lie algebra consisting of trace-free 2×2 matrices. The basis for the simple Lie algebra is

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{1}$$

with commutator properties

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1. \tag{2}$$

To generate the hierarchy, we use the following matrix loop algebra:

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \left\{ \sum_{i\geq 0} A_i \lambda^{n-i} | A_i \in \mathrm{sl}(2,\mathbb{R}), i \geq 0, n \in \mathbb{Z} \right\}. \tag{3}$$

In particular, the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ contains elements of the form $\lambda^m e_1 + \lambda^l e_2 + \lambda^p e_3$ with arbitrary integers m, l, p. Many well-known soliton hierarchies are generated from the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ [5–11].

In this paper, we will introduce a new spectral matrix and explain why it generalizes the D-Kaup–Newell spectral matrix. We then generate its soliton hierarchy. Next, we apply the trace identity to engender the Hamiltonian structure and discuss why the hierarchy is Liouville integrable, i.e., the hierarchy has infinitely many commuting symmetries and conserved functionals. We present a reduction of the spectral matrix to produce a completely different soliton hierarchy which is shown to have bi-Hamiltonian structure. Lastly, we discuss a few ideas for further research associated with the Lie algebra $\mathrm{so}(3,\mathbb{R})$.

2. A generalized D-Kaup–Newell spectral problem

Let us introduce a spectral matrix:

$$U = U(u,\lambda) = (\lambda^2 - r)e_1 + (\lambda p + s)e_2 + (\lambda q + v)e_3 = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix}, \tag{4}$$

and consider the following isospectral problem:

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix}\phi, \ U \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \ u = \begin{bmatrix} p \\ q \\ r \\ s \\ v \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{5}$$

where p, q, r, s, and v are potentials.

The D-Kaup–Newell spectral problem is known [7] to be

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix}\phi, \quad U \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{6}$$

which depends on three potentials: p, q, and r. The new spectral problem (5) is a generalization of the D-Kaup–Newell spectral problem adding two new potentials s and v. Previously, the cases $r = \alpha$ and $r = \alpha pq$, where α is a constant, have been shown to generate integrable hierarchies [12,13] for the D-Kaup–Newell spectral problem (6).

3. The soliton hierarchy

We assume a solution to the stationary zero curvature equation, $W_x = [U, W]$, to be of the form

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \tag{7}$$

and we get the equations

$$\begin{cases} a_x = -qb\lambda + pc\lambda - vb + sc, \\ b_x = -2pa\lambda + 2b\lambda^2 - 2sa - 2rb, \\ c_x = 2qa\lambda - 2c\lambda^2 + 2va + 2rc. \end{cases} \tag{8}$$