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Splitting methods for differential approximations of the radiative transfer equation

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ABSTRACT

The radiative transfer equation (RTE) has wide applications in sciences and engineering. Due to high dimensionality and integro-differential nature, the equation is difficult to solve numerically. In the literature, several approximation methods for solving the RTE numerically have been developed. Among them, a family of differential approximations of RTE, the so-called RT/DAE was proposed. In this paper, we establish a framework of the splitting method for RT/DAE and provide convergence analysis. We introduce the classic source iteration method, compare it with the new splitting method and prove the splitting method has superior convergence properties. Finally, we provide numerical examples demonstrating the effectiveness of the splitting method.

1. Introduction

As a forward model to describe the propagation or radiation of particles inside a medium, the radiative transfer equation (RTE) has wide applications in such as physics, chemistry, and other areas of sciences and engineering, see [10,12–16,20,21]. Let X be a domain in $\mathbb{R}^3$ with a Lipschitz boundary ∂X. The unit outward normal $\boldsymbol{n}(\boldsymbol{x})$ exists a.e. on ∂X. Denote by Ω the unit sphere in $\mathbb{R}^3$. The radiative transfer equation is given by

$$\boldsymbol{\omega}\cdot\nabla u+\mu_t u=\mu_s Su+f \quad \text{in } X\times\Omega. \tag{1.1}$$

Here the unknown function u depends on a spatial variable $\boldsymbol{x}\in X$ and an angular variable $\boldsymbol{\omega}\in\Omega$. $\boldsymbol{\omega}\cdot\nabla u$ denotes the generalized directional derivative of u in the direction $\boldsymbol{\omega}$. For the spherical coordinate system,

$$\boldsymbol{\omega}=(\sin\theta\cos\psi,\sin\theta\sin\psi,\cos\theta)^T,\quad 0\le\theta\le\pi,\quad 0\le\psi\le 2\pi.$$

$\mu_t=\mu_a+\mu_s$, μ_a is the macroscopic absorption cross section, μ_s is the macroscopic scattering cross section, and f is a source function in X. The integral operator S is defined by

$$Su(\boldsymbol{x},\boldsymbol{\omega})=\int_\Omega k(\boldsymbol{\omega}\cdot\hat{\boldsymbol{\omega}})u(\boldsymbol{x},\hat{\boldsymbol{\omega}})\,\mathrm{d}\sigma(\hat{\boldsymbol{\omega}})$$

with k a nonnegative normalized phase function:

$$\int_\Omega k(\boldsymbol{\omega}\cdot\hat{\boldsymbol{\omega}})\,\mathrm{d}\sigma(\hat{\boldsymbol{\omega}})=1\quad\forall\boldsymbol{\omega}\in\Omega.$$

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For the Henyey–Greenstein phase function [9],

$$k(t) = \frac{1-g^2}{4\pi(1+g^2-2gt)^{3/2}}, \quad t \in [-1,1],$$

where the parameter $g \in (-1,1)$ is the anisotropy factor of the scattering medium. Note that $g = 0$ for isotropic scattering, $g > 0$ for forward scattering and $g < 0$ for backward scattering.

Eq. (1.1) is supplemented by the boundary condition

$$u = u_{\text{in}} \quad \text{on } \Gamma_-, \tag{1.2}$$

where $\Gamma_- = \{(\boldsymbol{x},\boldsymbol{\omega}) | x \in \partial X, \boldsymbol{\omega} \in \Omega, \boldsymbol{\omega}\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$ is the incoming boundary.

We make the following assumptions on the data

$$\mu_t, \mu_s \in L^\infty(X), \quad \mu_s \geq 0,\ \mu_t - \mu_s \geq c_0 > 0 \text{ a.e. in } X, \tag{1.3}$$

$$f \in L^2(X\times\Omega), \quad u_{\text{in}} \in L^2(\Gamma_-). \tag{1.4}$$

It is well-known [1] that the boundary value problems (1.1) and (1.2) has a unique solution in the space

$$V := \left\{ v \in L^2(X\times\Omega) \mid \boldsymbol{\omega}\cdot\nabla v \in L^2(X\times\Omega), v|_{\Gamma_-} \in L^2(\Gamma_-) \right\},$$
$$V_0 := \{ v \in V \mid v = 0 \text{ on } \Gamma_- \}.$$

In the following, we use $(\cdot,\cdot)$ and $\|\cdot\|$ for the standard inner product and norm in the space $L^2(X\times\Omega)$.

The RTE is a high-dimensional problem with five independent variables. It can be viewed as a hyperbolic-type integrodifferential equation involving both partial derivatives and integrals. Because of the high dimensionality and tight coupling between variables, it is challenging to develop effective numerical methods to solve the RTE. When the RTE is discretized using the discrete-ordinate method [2], the integral term Su is approximated by a summation that involves all the numerical integration points on the unit sphere. Consequently, for the resulting discrete system, each degree of freedom is coupled not only to degrees of freedom representing nearby nodes in space, but also to all other degrees of freedom representing the same spatial location with different angle. As a result the discrete system is relatively dense, and many of the methods for solving sparse systems from discretization of partial differential equations cannot be applied efficiently. Therefore, various approximation methods have been proposed for the RTE, including the simplified spherical harmonics method [6], the delta-Eddington approximation [17], the Fokker–Planck approximation [18,22], the generalized Fokker–Planck approximation [19], the Boltzmann–Fokker–Plank approximation [8,23], the Fokker–Planck–Eddington approximation and the generalized Fokker–Planck–Eddington approximation [11].

In [4], a family of differential approximations of the RTE, the so-called RT/DAEs is studied. The RT/DA equation is based on the approximation of the integral operator S by a sequence of linear combinations of the inverse of linear elliptic differential operators on the unit sphere. For a spherical harmonic of order n, $Y_n(\boldsymbol{\omega})$ (cf. [7] for an introduction),

$$(SY_n)(\boldsymbol{\omega}) = k_n Y_n(\boldsymbol{\omega}),$$

where

$$k_n = 2\pi \int_{-1}^{1} k(s) P_n(s)\, \mathrm{d}s$$

and P_n is the Legendre polynomial of degree n. Therefore k_n is an eigenvalue of S with spherical harmonics of order n as the corresponding eigenfunctions. It can be proved that $k_n \geq 0$ [5], and moreover, since $|P_n| \leq 1$ in $[-1,1]$, we have that

$$k_n \leq 2\pi \int_{-1}^{1} k(s)\, ds = 1.$$

For the Henyey–Greenstein phase function, we have

$$k_n = g^n, \quad n = 0, 1, \ldots.$$

Throughout this paper an RT/DA equation with j terms for the approximation of the integral operator will be called an RT/DA$_j$ equation. Consider a j-term RT/DAE:

$$\boldsymbol{\omega}\cdot\nabla u + \mu_t u = \mu_s S_j u + f, \tag{1.5}$$

where

$$S_j = \sum_{i=1}^{j} \lambda_{j,i} (I - \alpha_{j,i}\Delta^*)^{-1}.$$

The operator Δ^* is the Laplace–Beltrami operator on the unit sphere. In sphereical coordinates it is defined as

$$\Delta^* u = \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \psi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right).$$