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On the dot product of graphs over monogenic semigroups

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ABSTRACT

Now define S a cartesian product of finite times with S_M^n which is a finite semigroup having elements $\{0, x, x^2, \ldots, x^n\}$ of order n. $\Gamma(S)$ is an undirected graph whose vertices are the nonzero elements of S. It is a new graph type which is the dot product. k be finite positive integer for $0 \leq \{i_t\}_{t=1}^k, \{j_t\}_{t=1}^k \leq n$, any two distinct vertices of S $(x^{i_1}, x^{i_2}, \ldots, x^{i_k})$ and $(x^{j_1}, x^{j_2}, \ldots, x^{j_k})$ are adjacent if and only $(x^{i_1}, x^{i_2}, \ldots, x^{i_k}) \cdot (x^{j_1}, x^{j_2}, \ldots, x^{j_k}) = 0_{S_M^n}$ (under the dot product) and it is assumed $x^{i_t} = 0_{S_M^n}$ if $i_t = 0$.

In this study, the value of diameter, girth, maximum and minimum degrees, domination number, clique and chromatic numbers and in parallel with perfectness of $\Gamma(S)$ are elucidated.

1. Introduction and preliminaries

Historically, Beck had started studying about connections between graph theory and ring theory, especially for commutative rings, in 1988. He is also interested in coloring of commutative rings, in [6]. This is continued by adding the zero divisor and the dot product graphs. In [3], Anderson and Livingston had studied about the same relation for zero divisor graphs. They denoted the zero divisor graph of a commutative ring by adding the zero elements of a commutative ring. Commutative semigroups with nonzero zero divisors are added to zero divisor graphs, see [12–14]. Anderson and Badawi continued their study which is about the zero divisor graph of commutative rings, in [4]. The zero-divisor graph is studied by a lot of authors in [1,2,7,19]. Akgüneş has defined the graph of monogenic semigroups and found sharp and strict results for some parameters in graph theory, see [9]. At last the dot product graph has been studied by Badawi, in [5], for commutative rings. He interested in both graphs $TD(R)$ and $ZD(R)$. He obtained some results for commutative rings by using an integral domain for a ring R with the help of graph theory. Also there are lots of studies about the graph parameters and zero-divisor graphs of commutative rings, see [2,10,11,17,18].

So we are interested in this study about the dot product graph and monogenic semigroups.

Let S be a cartesian product of k times S_M^n.

$$S_M^n = \{0, x, x^2, \ldots, x^n\}$$

and

$$S = S_M^n \times S_M^n \times \cdots \times S_M^n \text{ for } k \text{ times } 1 \leq k < \infty \tag{1}$$

Any two non-zero elements of S will be denoted by $X = (x^{i_1}, x^{i_2}, \ldots, x^{i_k})$ and $Y = (x^{j_1}, x^{j_2}, \ldots, x^{j_k})$ for $\{i_t\}_{t=1}^k, \{j_t\}_{t=1}^k \in \{0, 1, 2 \ldots, n\}$ where $x^{i_t} = 0_{S_M^n}$ if and only if $i_t = 0$.

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Let us define the dot product function among elements of S.

$$X \cdot Y = \left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right) \cdot \left(x^{j_1}, x^{j_2}, \ldots, x^{j_k}\right)$$
$$= x^{i_1}.x^{j_1} + \ldots + x^{i_k}.x^{j_k} = x^{i_1+j_1} + \ldots + x^{i_k+j_k}$$

Then it can be defined that the dot product graph $\Gamma(S)$ is an (undirected) graph with vertices X, $Y \in S^* = S/\{0_S\}$ such that X and Y are adjacent if $X \cdot Y = 0_{S_M^n}$ and we say that the vertices X and Y are adjacent. This is denoted by $X \sim Y$ if their dot product is equal to the zero element of S_M^n.

It is obtained that $\Gamma(S)$ is always connected and its diameter is equal to 2 ($diam(\Gamma(S)) = 2$) and its girth is equal to 3 ($Gith(\Gamma(S)) = 3$). Then we find a way to calculate the maximum degree and minimum degree of $\Gamma(S)$ which is connected with n and k. We observe that the domination number is 1. Then we accept that k is 2 to find the clique and chromatic number of $\Gamma(S)$. So it is defined that whether $\Gamma(S)$ is perfect graph or not.

2. Some properties of $\Gamma(S)$

First of all, let us see how any two nonzero elements of S are adjacent. It is denoted that any two nonzero distinct vertices of $\Gamma(S)$ $X = \left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right)$ and $Y = \left(x^{j_1}, x^{j_2}, \ldots, x^{j_k}\right)$ for $0 \le \{i_t\}_{t=1}^k, \{j_t\}_{t=1}^k \le n$ are adjacent if $X \cdot Y = 0$, in other words $x^{i_1+j_1} + x^{i_2+j_2} + \cdots + x^{i_k+j_k} = 0$. This equality is provided by if $i_t + j_t > n$ or i_t is equal to the zero element of S_M^n or j_t is equal to zero element of S_M^n for each element $t \in \{1, 2, \ldots, k\}$. In other words; $\left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right) \sim \left(x^{j_1}, x^{j_2}, \ldots, x^{j_k}\right)$ iff $(i_t + j_t > n)$ or $(i_t = 0)$ or $(j_t = 0)$, $\forall t \in \{1, 2, \ldots, k\}$.

Now we can look at the following definition for the first observation of this study.

The *diameter* of a graph G is defined as the set $diam(G) = \sup\{d(x, y) : x \text{ and } y \text{ are vertices of } G\}$.

Theorem 2.1. *For any S, as defined in (1),*

$$diam(\Gamma(S)) = 2.$$

Proof. Let $\left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right)$, $\left(x^{j_1}, x^{j_2}, \ldots, x^{j_k}\right)$ for $0 \le \{i_t\}_{t=1}^k, \{j_t\}_{t=1}^k \le n$ be two distinct and nonadjacent vertices of $\Gamma(S)$ and different from $(x^n, x^n, \ldots, x^n)$. There are two cases for i_t. If $i_t = 0$ then $x^{i_t}.x^n = 0_{S_M^n}$ since $x^{i_t} = 0_{S_M^n}$. On the contrary, $i_t \neq 0$ then $x^{i_t}.x^n = 0_{S_M^n}$ since $i_t + n > n$. Thus $\left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right)$ is adjacent to $(x^n, x^n, \ldots, x^n)$. Likewise $\left(x^{j_1}, x^{j_2}, \ldots, x^{j_k}\right)$ is adjacent to $(x^n, x^n, \ldots, x^n)$, too. Hence the distance between any two distinct vertices is equal to 2.

$$diam(\Gamma(S)) = 2.$$

□

The *girth* of a graph is the length of the shortest cycle contained in that graph. If the graph does not contain any cycles, then its girth is defined to be infinity.

Theorem 2.2. *The girth of $\Gamma(S)$, as defined (1), is equal to 3.*

$$Girth(\Gamma(S)) = 3.$$

Proof. At first, we know that $(x^n, x^n, \ldots, x^n)$ is adjacent to $\left(x^2, x^2, \ldots, x^2\right)$ since $n + 2 > n$. Then $\left(x^2, x^2, \ldots, x^2\right)$ and $\left(x^{n-1}, x^{n-1}, \ldots, x^{n-1}\right)$ are adjacent vertices since $n - 1 + 2 = n + 1 > n$. Finally $\left(x^{n-1}, x^{n-1}, \ldots, x^{n-1}\right)$ is adjacent to $(x^n, x^n, \ldots, x^n)$ since $n - 1 + n = 2n - 1 > n$. Thus we have a cycle whose length is equal to 3 over the dot product graph of S_M^n. So we get;

$$Girth(\Gamma(S)) = 3.$$

□

It is known that the *maximum degree* of G is the largest degree in G denoted by Δ and the *minimum degree* of G is the smallest degree in G (see, [15]).

Theorem 2.3. *Let S_M^n be a monogenic semigroup and S be a cartesian product of finite times S_M^n. Then*

$$\Delta(\Gamma(S)) = (n+1)^k - 2 \text{ and } \delta(\Gamma(S)) = 2^k - 1.$$

Proof. First, let us choose any vertex of S for the maximum degree of $\Gamma(S)$ such as $\left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right)$ to be adjacent to $(x^n, x^n, \ldots, x^n)$ for $0 \le \{i_t\}_{t=1}^k \le n$. So $x^{i_t}.x^n = 0_{S_M^n}$ for each elements $i_t \in \{0, 1, 2, \ldots, n\}$ then $\left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right)$ is equal to $(n+1)^k$ different vertices. However, it can not be equal to zero element of S and $(x^n, x^n, \ldots, x^n)$. Hence there are $(n+1)^k - 2$ different vertices which are adjacent to $(x^n, x^n, \ldots, x^n)$. It means the maximum degree of $\Gamma(S)$ is equal to $(n+1)^k - 2$.

For the minimum degree of $\Gamma(S)$, we obtain the adjacency vertices of $(x, x, \ldots, x)$ since it has minimum exponent elements of S_M^n. Let any vertex of S $\left(x^{i_1}, x^{i_2}, \ldots, x^{i_k}\right)$ be adjacent to $(x, x, \ldots, x)$ for $\{i_t\}_{t=1}^k \in \{0, 1, 2, \ldots, n\}$. So either $1 + i_t > n$ or $i_t = 0$ have to be provided according to each elements $t \in \{1, 2, \ldots, k\}$ since $x.x^{i_t} = 0_{S_M^n}$. Hence it is equal to either n or 0 for