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Completing a well-balanced numerical method for a model of two-phase flows by computing correctors

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ABSTRACT

We complete a well-balanced numerical method by introducing computing correctors to an earlier scheme for a model of two-phase flows. Each improvement based on a corrector to the scheme is designed to reduce the size of the errors across the interface of each node when using the solid contact to absorb the nonconservative terms. Three correctors of two kinds are presented. One corrector of the first kind is designed to correct the states on both side of the solid contact at each node and the corresponding numerical flux before applying the iterative scheme. Two correctors of the second kind are designed to correct the state given by the iterative scheme depending on the sign of the velocity of the solid contact. These improvements are still well-balanced schemes. Tests show that the improvement by using the corrector of the first kind gives relatively better results, and the improvements by using one corrector of the second kind give much better results. Interestingly, we find that improvements by using a corrector of second kind can resolve the accuracy problem of the existing scheme when its approximate solutions might converge to the solution slightly different from the exact solution.

1. Introduction

We are interested in numerical approximations for a model of two-phase flows, which is arisen from the modeling of the deflation-to-detonation transition in granular explosives, see [4,7]. Here, we simplify the model by assuming that the fluid in each phase is isentropic, and nonreactive. The governing equations form a nonconservative system of balance laws, see e.g. [12] for the mathematical formulation of nonconservative hyperbolic systems involving nonconservative terms.

Recently, a one-parameter family of fast and well-balanced numerical schemes was presented in [28]. Schemes in this family are constructed by first using the solid contact of the model to absorb the nonconservative terms at each node, and second using underlying conservative schemes. The underlying schemes are formed by using convex combinations of numerical fluxes of the first-order Lax–Friedrichs scheme g_{LF} and the second Richtmyer's scheme g_R. The CPU time of schemes in this family can be reduced when θ increases. In [28], three values of the parameter θ are tested: $0, 1/(1+CFL), CFL$, where CFL is the stability number, and it is shown that the scheme corresponding to $\theta = CFL$ can give the best results. Although larger values of θ may work and may give better results, we choose $\theta = CFL$ for simplicity, which defines a scheme labeled by Fast 3. Since the values on both sides of the solid contact at each node are used to replace the approximate states on the whole cells, errors are expected. This motivates us to "minimize" or at least, to reduce the size of this kind of errors

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by adding/subtracting a certain "residue". This will be done by introducing three correctors to the Fast 3 scheme as follows. The first corrector is designed to correct the states on both side of the solid contact at each node and the corresponding numerical flux by adding/subtracting a certain "residue" caused by using these states to replace the approximate states for the whole cells *before* applying the underlying scheme. Two other correctors are designed to correct the approximate state produced by the Fast 3 scheme by adding/subtracting a certain "residue" *directly* to this state. This task, as seen later, will depend on the sign of the velocity of the solid contact. These improvements are still well-balanced schemes. Numerical tests are designed for different locations and the sign of the velocity of the solid contact in Riemann solutions (see [25]): supersonic/subsonic region and positive/negative velocity. These tests show that the improvement by the corrector of the first kind gives slightly better results, and the improvements by using one corrector of the second kind give much better results. Furthermore, as expected Fast 3 scheme might give approximate solutions converging to a solution which is slightly different from the exact solution. It is very interesting that the improvements by using a corrector of second kind can resolve this accuracy problem: it provides approximate solutions which converge to the exact solution, as the order of convergence is stable. Schemes of this type seem to be more suitable for certain users than the ones building on exact solutions, see for example [11]. Indeed, because of the resonance phenomenon, the structure of exact solutions of the Riemann problem for two-phase flow models is often very complicated, see [25].

There are many works in the literature on the numerical approximations of nonconservative systems. Well-balanced schemes for a single conservation law with a source term are studied in [3,5,6,16,17]. A robust entropy-satisfying finite volume scheme for the isentropic Baer–Nunziato model was constructed in [9]. Relaxation schemes for two-phase flow models were presented in [1,20]. Godunov-type schemes for two-phase flow models were considered in [11,23,24]. A modeling of two-phase flows using a two-fluid and two-pressure assumption was presented in [14]. Other numerical schemes for various models of multi-phase flows were considered in [21,22,26,27,29]. Well-balanced schemes for other nonconservative hyperbolic models were considered in [3,8,10,18]. Theoretical studies for various nonconservative hyperbolic models have been presented in [2,15,19,25]. See also the references therein.

The organization of this paper is as follows. In Section 2 we present basic concepts and terminologies of the model. Section 3 is devoted to the construction of the correctors and the improvements of the well-balanced Fast 3 scheme. Section 4 provides us with numerical tests for different cases depending on the location and the sign of the velocity of the solid contact of the exact Riemann solution: the location of the solid contact in the supersonic/subsonic region with a positive/negative velocity. Finally, Section 5 is devoted to conclusions and discussions.

2. Backgrounds

2.1. Nonstrict hyperbolicity

The governing equations of the model under consideration are given by

$$\begin{aligned}
&\partial_t(\alpha_g\rho_g) + \partial_x(\alpha_g\rho_g u_g) = 0,\\
&\partial_t(\alpha_g\rho_g u_g) + \partial_x(\alpha_g(\rho_g u_g^2 + p_g)) = p_g\partial_x\alpha_g,\\
&\partial_t(\alpha_s\rho_s) + \partial_x(\alpha_s\rho_s u_s) = 0,\\
&\partial_t(\alpha_s\rho_s u_s) + \partial_x(\alpha_s(\rho_s u_s^2 + p_s)) = -p_g\partial_x\alpha_g,\\
&\partial_t\rho_s + \partial_x(\rho_s u_s) = 0, \quad x \in R, t > 0.
\end{aligned} \tag{2.1}$$

In (2.1), the subscripts g and s indicate the quantities in the g-phase, which may be referred to as the gas phase, and in the solid phase, which may be referred to as the solid phase, though this work can cover other materials such as liquids. The notations $\alpha_k, \rho_k, u_k, p_k, k = g, s,$ denote the volume fraction, density, velocity, and pressure in the k-phase, $k = g, s,$ respectively. The volume fractions satisfy

$$\alpha_s + \alpha_g = 1. \tag{2.2}$$

The model (2.1) can be re-written as a nonconservative system

$$U_t + A(U)U_x = 0, \tag{2.3}$$

where

$$U = \begin{pmatrix} \rho_g \\ u_g \\ \rho_s \\ u_s \\ \alpha_g \end{pmatrix}, \qquad A(U) = \begin{pmatrix} u_g & \rho_g & 0 & 0 & \dfrac{\rho_g(u_g - u_s)}{\alpha_g} \\ \dfrac{p_g'(\rho_g)}{\rho_g} & u_g & 0 & 0 & 0 \\ 0 & 0 & u_s & \rho_s & 0 \\ 0 & 0 & \dfrac{p_s'(\rho_s)}{\rho_s} & u_s & \dfrac{p_g - p_s}{\alpha_s\rho_s} \\ 0 & 0 & 0 & 0 & u_s \end{pmatrix}. \tag{2.4}$$