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Local antimagic labeling of graphs☆

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ABSTRACT

A k-labeling of a graph G is an injective function ϕ from $E(G)$ to $m+k$ real numbers, where $m=|E(G)|$. Let $\mu_G(v)=\sum_{uv\in E(G)}\phi(uv)$. A graph is called *antimagic* if G admits a 0-labeling with labels in $\{1,2,\ldots,|E(G)|\}$ such that $\mu_G(u)\neq\mu_G(v)$ for any pair $u,v\in V(G)$. A well-known conjecture of Hartsfield and Ringel states that every connected graph other than K_2 admits an antimagic labeling. Recently, two sets of authors Arumugam, Premalatha, Bača, Semaničová-Fenŏvčíková, and Bensmail, Senhaji, Lyngsie independently introduced the weaker notion of a local antimagic labeling, which only distinguishes adjacent vertices by sum with labels in $\{1,2,\ldots,|E(G)|\}$. Both sets of authors conjecture that any connected graph other than K_2 admits a local antimagic labeling. In this paper, we prove that every subcubic graph without isolated edges admits a local antimagic labeling with $|E(G)|$ positive real labels. We also prove that each graph G without isolated edges admits a local antimagic k-labeling, where $k=\min\{\Delta(G)+1,\frac{3col(G)+3}{2}\}$, and $col(G)$ is the coloring number of G. Actually, the latter result holds for the list version.

1. Introduction

All graphs considered in this paper are finite and simple (loops and multiple edges are not allowed). We follow [6] for the terminology and notation not defined here. Let G be a graph. The least number s such that G has a vertex permutation in which each vertex is preceded by fewer than s of its neighbors is called the *coloring number* of G, denoted by col(G). If a graph G has no isolated edges, then we call it *nice*.

A *k-labeling* of a graph G is an injective function ϕ from $E(G)$ to a label set S of $m+k$ real numbers, where $m=|E(G)|$. Let $\mu_G(v)=\sum_{uv\in E(G)}\phi(uv)$. A graph is called *antimagic* if G admits a 0-labeling with labels in $S=\{1,2,\ldots,|E(G)|\}$ such that $\mu_G(u)\neq\mu_G(v)$ for any pair $u,v\in V(G)$. In 1990, Hartsfield and Ringel proposed the following conjecture:

Conjecture 1.1 *[10]*. *(Antimagic graph conjecture) Every connected nice graph is antimagic.*

There have been significant progresses toward Conjecture 1.1. Let G be a nice graph with n vertices. In 2004, Alon et al. [2] showed that there exist a constant c such that if G has minimum degree at least $c\cdot\log n$, then G is antimagic. They also proved that G is antimagic when the maximum degree of G is at least $n-2$. The latter result of Alon et al. was

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improved by Yilma [18] in 2013. Besides the above results on dense graphs, Conjecture 1.1 has been also verified for regular graphs, see [4,7,8].

Notably, however, Conjecture 1.1 is still unsolved even for some particularly simple natural graph classes such as trees. Recently, two sets of authors Arumugam, Premalatha, Bača, Semaničová-Fenõvčíková [3], and Bensmail, Senhaji, Lyngsie [5] independently introduced a weaker notion. A k-labeling is called *local k-antimagic labeling* if for every $uv \in E(G)$, we have that $\mu_G(u) \neq \mu_G(v)$. A graph G is called *local antimagic* if G has a local 0-antimagic labeling.

Conjecture 1.2 [3,5]**.** *Every nice connected graph G is local antimagic with the label set* $S = \{1, 2, \ldots, |E(G)|\}$.

Arumugam et al. [3] were motivated by using the vertex sums to define a proper coloring of G. While Bensmail et al. [5] were motivated by the 1-2-3 Conjecture, which means that the edges of G are weighted by {1, 2, 3} such that $\mu_G(u) \neq \mu_G(v)$ for any $uv \in E(G)$. The latest result about the 1-2-3 Conjecture is that weight set {1, 2, 3, 4, 5} suffices [13]. Conjecture 1.2 is true for nice trees [5] and nice paths, nice regular graphs, wheels, nice complete multipartite graphs [9]. Hu et al. proved that each degenerate graph admits a local antimagic orientation [12]. In [5] Bensmail et al. verified that subcubic graphs admits a local 6-antimagic labeling with $S = \{1, 2, \ldots, |E(G)| + 6\}$, every nice 2-degenerate graph admits a local 4-antimagic labeling with $S = \{1, 2, \ldots, |E(G)| + 4\}$, and every nice graph admits a local k-antimagic labeling with $S = \{1, 2, \ldots, k\}$, where $k = \min\{|E(G)| + 2\Delta(G), 2|E(G)|\}$.

Obviously, nice paths, nice cycles, complete graphs K_n ($n \geq 3$) are local antimagic. In this paper, we improve the previous bounds in [5] and verify the following three theorems.

Theorem 1.1. *Assume that G is a nice connected subcubic graph, and S is a label set with |E(G)| positive real numbers. Then G is local antimagic with labels in S.*

Theorem 1.2. *Let G be a nice graph with maximum degree* $\Delta(G)$*. If all the vertices have odd degrees in G, then G is local* $(\Delta(G) + 1)$*-antimagic. Otherwise, G is local* $\Delta(G)$*-antimagic.*

Theorem 1.3. *If G is a nice graph, then G is local* $(\frac{3col(G)+3}{2})$*-antimagic.*

It is well known that every planar graph is a 5-degenerate graph. Thus the following corollary is obvious.

Corollary 1.1. *Let G be a nice planar graph, then G is local 11-antimagic.*

2. Preliminaries

To prove our main results, we need to introduce some notations and several lemmas.

The degree of a vertex v in a graph G is denoted by $d_G(v)$. A vertex v of degree l (at least l, at most l) is called an *l-vertex* (*l^+-vertex*, *l^--vertex*, respectively). Let $N_G(v)$ be the set of all the neighbors of v in G.

Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables, where $n \geq 1$. By $c_P(x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n})$, we denote the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ in $P(x_1, x_2, \ldots, x_n)$, where k_i $(1 \leq i \leq n)$ is a non-negative integer.

Lemma 2.1 [1]**.** *(Combinatorial Nullstellensatz) Let* $\mathbb{F}$ *be an arbitrary field, and let* $P = P(x_1, \ldots, x_n)$ *be a polynomial in* $\mathbb{F}[x_1, \ldots, x_n]$*. Suppose the degree deg(P) of P equals* $\sum_{i=1}^{n} k_i$*, where each* k_i *is a non-negative integer, and suppose the coefficient of* $x_1^{k_1} \ldots x_n^{k_n}$ *in P is non-zero. If* $S_1, \ldots, S_n$ *are subsets of* $\mathbb{F}$ *with* $|S_i| > k_i$*, then there are* $s_1 \in S_1, \ldots, s_n \in S_n$ *so that* $P(s_1, \ldots, s_n) \neq 0$*.*

Lemma 2.2 [14]**.** *Let* B_1, B_2 *be the sets of integers, with* $|B_1| = m \geq 2$ *and* $|B_2| = n \geq 2$*. Let* $B_3 = \{x + y \mid x \in B_1, y \in B_2, x \neq y\}$*. Then* $|B_3| \geq m + n - 3$*. Moreover, if* $B_1 \neq B_2$*, then* $|B_3| \geq m + n - 2$*. Meanwhile, among the pairs in* B_3*, there are at most two pairs satisfying* $x - y = z$*, where z is a constant.*

Lemma 2.3. *Let* $S_1, \ldots, S_\lambda$ *be strictly increasing sequences of* $\mathbb{R}$ *with* $|S_i| = m_i$*, i.e.* $S_i = \{s_i^1, s_i^2, \ldots, s_i^{m_i}\}$ *and* $s_i^j < s_i^k$ *for any* $1 \leq j < k \leq m_i$*. Given some inequalities below:*

$$\begin{cases} a_1 x_1 + a_{12} x_2 + \cdots + a_{1\lambda} x_\lambda \neq c_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2\lambda} x_\lambda \neq c_2 \\ \quad \vdots \\ a_{t1} x_1 + a_{t2} x_2 + \cdots + a_{t\lambda} x_\lambda \neq c_t, \end{cases} \tag{1}$$

where a_{ij}*,* c_i *are some real numbers, and in each inequality there are exactly two non-zero* a_{ij} *for* $1 \leq i \leq t$*,* $1 \leq j \leq \lambda$*. Let* r_i^1 *be the number of inequalities, in which it holds that* $a_{ki} \neq 0$ $(1 \leq k \leq t)$*, and there exists some j with* $1 \leq j \leq i - 1$*, such that* $a_{kj} \neq 0$*, where* $1 \leq k \leq t$*. Let* r_i^2 *be the number of inequalities, in which it holds that* $a_{ki} \neq 0$ $(1 \leq k \leq t)$*, and there exists some j with* $i + 1 \leq j \leq \lambda$*, such that* $a_{kj} \neq 0$*, where* $1 \leq k \leq t$*. Assume that* $r_i = r_i^1 + r_i^2$ *for* $1 \leq i \leq \lambda$*. Then we can find a set* $S = \{(x_1, \ldots, x_\lambda) \mid x_i \in S_i, i \in [\lambda]\}$ *satisfying all the inequalities in (1) such that* $|S'| = \sum_{i=1}^{\lambda}(m_i - r_i) - (\lambda - 1)$*, where* $S' = \{\sum_{i=1}^{\lambda} x_i \mid (x_1, \ldots, x_\lambda) \in S\}$*.*