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Two-dimensional shifted Legendre polynomials operational matrix method for solving the two-dimensional integral equations of fractional order

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This work approximates the unknown functions based on the two-dimensional shifted Legendre polynomials operational matrix method (2D-SLPOM) for the numerical solution of two-dimensional fractional integral equations. The present method reduces these equations to a system of algebraic equations and then this system will be solved numerically by Newton's method. Moreover, an estimation of the error bound for this algorithm will be shown by preparing some theorems. Some examples are presented to demonstrate the validity and applicability of the proposed method with respect to the two-dimensional block pulse functions method (2D-BPFs) and two-dimensional Bernstein polynomials operational matrix method (2D-BPOM).

1. Introduction

Applications of fractional calculus within various areas of science, engineering, bio-engineering, finance, viscoelasticity, electrochemistry, control and electromagnetic, applied mathematics and others become nowadays wide and flourishing [1,2]. For this reason, fractional differential equations have been considered by many authors and some numerical methods for their solutions have been proposed. For example Song and Wang used the Adomian decomposition method and its application to fractional differential equations [3]. In [4], Chebyshev spectral method based on operational matrix was used for initial and boundary value problems of fractional order. In [5], fractional partial differential equations with variable coefficients were solved using the reconstruction of variational iteration method. Reconstruction of variational iteration method was used to solve multi-order fractional differential equations [6]. Authors in [7] solved fractional nonlinear Volterra integro-differential equations by the second kind Chebyshev wavelets. In [8], authors implemented hybrid collocation method for solving fractional integro-differential equations. In [9], Chebyshev pseudo-spectral method was used for solving the linear and nonlinear systems of fractional integro-differential equations.

Recently, a lot of attentions has been devoted to the study of Legendre polynomials to investigate various scientific models. Using these polynomials made it possible to solve differential equations of Lane–Emden type [10], second and fourth order equations [11], Cahn–Hilliard equations with Neumann boundary conditions [12], Fredholm integral [13], Helmholtz equation [14], second kind Volterra integral equations [15], high-order linear Fredholm integro-differential [16], fractional differential equations [17] and Abel's integral equation [18]. In [19], the second Legendre polynomials have been used to

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solve Volterra–Fredholm integral equations. Linear Fredholm integral equations of the second kind have been solved by using hybrid Legendre-Block-Pulse functions in [21]. In [22], Mokhtary and Ghoreishi, proved the L^2 convergence of Legendre Tau method for numerical solution of nonlinear fractional integro-differential equations.

One of the most important subdivision of fractional calculus is the two-dimensional fractional integral equations which will be studied in this paper. Recently, some numerical methods have been used to solve these equations by researchers. Two-dimensional fractional percolation equation was solved in [23]. In [24], the two-dimensional time-fractional wave equation has been solved by using the Homotopy perturbation method. Two-dimensional fractional sub-diffusion equation was solved by using the orthogonal spline collocation method [25]. In [26], analytical approximations of the two and three dimensional time-fractional telegraphic equations have been done using the reduced differential transform method.

In this paper, we will present an efficient numerical method for solving the two-dimensional fractional integral equations. For this purpose we will use two-dimensional shifted Legendre polynomials operational matrix method (2D-SLPOM) to solve the two-dimensional Volterra integral equations of fractional order in the following form

$$u(x,y)-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y\frac{k(x,y,\zeta,\xi)[u(\zeta,\xi)]^p}{(x-\zeta)^{1-\alpha_1}(y-\xi)^{1-\alpha_2}}d\xi d\zeta=f(x,y), \tag{1}$$

and the two-dimensional Fredholm integral equations of fractional order

$$u(x,y)-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{L_1}\int_0^{L_2}\frac{k(x,y,\zeta,\xi)[u(\zeta,\xi)]^p}{(L_1-\zeta)^{1-\alpha_1}(L_2-\xi)^{1-\alpha_2}}d\xi d\zeta=f(x,y), \tag{2}$$

where the functions $k(x, y, \zeta, \xi)$ and $f(x, y)$ are known for $(x, y)\in[0, L_1]\times[0, L_2]$ and $u(\zeta, \xi)$ is the unknown function to be determined, also $p\geq 1$ is a positive integer.

Recently, Eqs. (1) and (2) have been solved using 2D-BPFs and 2D-BPOM (see [27,33]). In this paper, we will extend 2D-SLPOM to approximate the solution of Eqs. (1) and (2). The properties of this method are used to reduce the problem to a system of algebraic equations. Besides, an estimation of error bound for this method will be given. Finally, we apply this method to several examples in order to show the efficiency of the presented method. The rest of this paper is organized as follows.

In Section 2, we will recall some preliminaries and properties of Legendre polynomials and a brief review of the fractional integral. Almost operational matrix is presented in Section 3. Section 4 presents approximate solution of the two-dimensional integration of fractional order via 2D-SLPOM. In Section 5, we give an error estimation for the presented method. Section 6 offers some numerical examples to illustrate the efficiency of this algorithm. Finally, concluding remarks are drawn in Section 7.

2. Preliminaries

2.1. Definition and properties of Legendre polynomials

In this section, some preliminaries and notations of Legendre polynomials which are necessary for later are recalled. The well-known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula (see [19]):

$$\phi_{i+1}(t)=\frac{2i+1}{i+1}t\phi_i(t)-\frac{i}{i+1}\phi_{i-1}(t),\quad i=1,\ 2,\ldots,$$

where $\phi_0(t)=1$ and $\phi_1(t)=t$. In order to use these polynomials on the interval $[a, b]$, we define the so-called shifted Legendre polynomials of degree i by introducing the change of variable $x=\frac{2}{b-a}t-\frac{b+a}{b-a}$ as follows (see [20]):

$$\Phi_i(x)=\phi_i\left(\frac{2}{b-a}t-\frac{b+a}{b-a}\right). \tag{3}$$

2D shifted Legendre polynomials are defined on $\Delta=[0,L_1]\times[0,L_2]$ as follows:

$$P_{ij}(x,y)=\phi_i\left(\frac{2}{L_1}t-1\right)\phi_j\left(\frac{2}{L_2}s-1\right),\ \ i,j=0,1,2,\ldots. \tag{4}$$

We consider the space $L^2(\Delta)$ equipped with the following inner product and norm:

$$\langle u(x,y),v(x,y)\rangle=\int_0^{L_1}\int_0^{L_2}u(x,y)v(x,y)dxdy,$$

$$\|u(x,y)\|=\langle u(x,y),u(x,y)\rangle^{\frac{1}{2}}=\left(\int_0^{L_1}\int_0^{L_2}|u(x,y)|^2dxdy\right)^{\frac{1}{2}}.$$

The set of 2D shifted Legendre polynomials forms a complete $L^2(\Delta)$-orthogonal system such that the orthogonality condition is

$$\int_0^{L_1}\int_0^{L_2}P_{ij}(x,y)P_{mn}(x,y)dxdy=\begin{cases}\dfrac{L_1L_2}{(2i+1)(2j+1)}, & \text{for } i=m,\ j=n,\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$