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Stability and convergence of compact finite difference method for parabolic problems with delay☆

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ABSTRACT

The compact finite difference method becomes more acceptable to approximate the diffusion operator than the central finite difference method since it gives a better convergence result in spatial direction without increasing the computational cost. In this paper, we apply the compact finite difference method and the linear θ-method to numerically solve a class of parabolic problems with delay. Stability of the fully discrete numerical scheme is investigated by using the spectral radius condition. When $\theta \in \left[0, \frac{1}{2}\right)$, a sufficient and necessary condition is presented to show that the fully discrete numerical scheme is stable. When $\theta \in \left[\frac{1}{2}, 1\right]$, the fully discrete numerical method is proved to be unconditionally asymptotically stable. Moreover, convergence of the fully discrete scheme is studied. Finally, several numerical examples are presented to illustrate our theoretical results.

1. Introduction

Delay differential equations provide a powerful means of modeling plenty of nature phenomena in scientific fields, such as ecology, biology, control systems, etc [1–4]. A main characteristic of the models is that the rate of variation of the quantity does not only depend on the present situation but also on the history.

When considering the applicability of numerical schemes for solving delay differential equations, it is important to analyze the stability of the methods. A widely used method is to apply the proposed numerical scheme to the following test equation

$$u_t = \lambda u + \mu u(t - \tau), \tag{1.1}$$

where λ and μ are parameters, and τ is the delay term. For example, Liu and Spijker [5] investigated stability of θ-method under the assumption $\mathrm{Re}(\lambda) < -|\mu|$. Guglielmi [6] further considered the delay-dependent stability of the θ-method. Zhao et al. [7] studied delay-dependent stability of boundary value methods when $\lambda, \mu \in \mathbb{R}$. Li and Zhang [8] developed some

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high-order Runge–Kutta methods which preserve delay-dependent stability of the test Eq. (1.1). More details about this can be found in an uncomplete list of references [9–16], the books [2,4] and so on.

When referring to some real-world problems, the space version should be taken into account. This research later yields the following typical reaction-diffusion equation with delay,

$$\frac{\partial}{\partial t}u(x,t) = c_1\frac{\partial^2}{\partial x^2}u(x,t) + c_2u(x,t-\tau), \tag{1.2}$$

where $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$ are parameters, and τ is the delay term. For example, Zubik-Kowal [17] studied the contractivity properties of the θ-methods. Huang and Vandewalle [18] studied delay-dependent stability of the PDEs with delay. Li et al. [19] introduced the compact finite difference methods to discrete the diffusion operator and considered long time stability of a fully discrete numerical scheme. Vandewalle and Gander [20] obtained convergence of overlapping Schwarz methods. Wu et al. [21] further presented sharper error estimates of overlapping Schwarz methods. For more details about the topic as well as their generations, we refer readers to [22–26] and the references therein.

The following model

$$\frac{\partial}{\partial t}u(x,t) = r_1\frac{\partial^2}{\partial x^2}u(x,t) + r_2\frac{\partial^2}{\partial x^2}u(x,t-\tau), \tag{1.3}$$

where $r_1 \in \mathbb{R}$ and $r_2 \in \mathbb{R}$ are parameters, is also used as the test equation and has also been extensively studied. In [27], Green and Stech obtained uniqueness, local existence, and global continuation of Eq. (1.3). Houwen and Sommeijer [28] discussed stability of some predictor–corrector schemes. Tian [29] studied the delay-independent stability of Euler method and Crank–Nicolson method. After that, Tian [30] considered asymptotic stability property of linear θ-method. Blanco-Cocom and Ávila-Vales [31] further considered stability and convergence of the linear θ-method for reaction-diffusion equations with delay. In [29–31], the diffusion operator was approximated by the standard second-order central finite difference method. Liang [32] introduced the Galerkin method to discrete the spatial variable and obtained stability and convergence of the fully discrete numerical method. Castro et al. [33] derived convergence of an explicit numerical method for solving the equation with variable coefficients in time.

In this study, we further investigate stability and convergence of a fully discrete numerical method for solving Problem (1.3) with the following initial and boundary conditions

$$\begin{aligned} &u(x,t) = u_0(x,t), \quad -\tau \le t \le 0, \quad x \in \Omega = [0,\pi],\\ &u(0,t) = u(\pi,t) = 0, \quad t \ge -\tau. \end{aligned}$$

The spatial discretization is performed by using the compact finite difference method, and the time discretization is achieved via the linear θ-method. By employing spectral radius condition, we discuss stability of the fully discrete numerical scheme. It is shown that when $\theta \in \left[0, \frac{1}{2}\right)$, the fully numerical scheme is asymptotically stable under the stepsize restriction. When $\theta \in \left[\frac{1}{2}, 1\right]$, the proposed numerical scheme is unconditionally stable. Then, convergence of the fully discrete numerical scheme is obtained. The proposed method has the order of $\mathcal{O}(\Delta x^4)$ in spatial direction. It gives better convergence result than the standard second-order central finite difference method without increasing the computational cost. Finally, several numerical experiments are performed to illustrate the theoretical results.

The rest of the paper is organized as follows. In Section 2, we propose the fully discrete numerical method for solving the test Eq. (1.3). In Section 3, we discuss stability and convergence of the proposed method. In Section 4, we give several numerical examples to illustrate our theoretical results. Finally, conclusions and discussions are summarized in Section 5.

2. The fully discrete numerical method

In this section, we present the fully discrete numerical method for the test Problem (1.3).

Let $\Delta t = \frac{\tau}{m}$ and $\Delta x = \frac{\pi}{N}$ be temporal and spatial stepsizes, respectively, where m and N are two positive integers. Define mesh points $t_k = k\Delta t$, $k = -m, -m+1, \ldots,$ $x_j = j\Delta x$, $j = 0, 1, \ldots, N$, $\Omega_\tau = \{t_k | k = -m, -m+1, \ldots\}$ and $\Omega_h = \{x_j | 0 \le n \le N\}$. Let u_j^k denote the numerical approximation of $u(x_j, t_k)$ and $\mathcal{V} = \{u_j^k | 0 \le j \le N, k \ge -m\}$ be grid function space defined on $\Omega_h \times \Omega_\tau$. For any grid function $u \in \mathcal{V}$, we will use the following notations:

$$\delta_t u_j^k = \frac{u_j^{k+1} - u_j^k}{\Delta t}, \quad \delta_x^2 u_j^k = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{\Delta x^2}, \quad u_j^{k+\frac{1}{2}} = \frac{1}{2}(u_j^k + u_j^{k+1}).$$

Let

$$\mathscr{A}_h u_j^k = \begin{cases} \dfrac{u_{j-1}^k + 10u_j^k + u_{j+1}^k}{12}, & j = 1, 2, \ldots, N-1,\\ u_j^k & j = 0, N. \end{cases} \tag{2.1}$$

Now, applying the compact finite difference method to discrete the diffusion operator and the linear θ-method to discrete the equation, we have