Applied Mathematics and Computation 322 (2018) 154–173

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Error analysis of projection methods for non inf-sup stable mixed finite elements. The transient Stokes problem

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ARTICLE INFO

Keywords:
Projection methods
PSPG stabilization
Non inf-sup stable elements

ABSTRACT

A modified Chorin–Teman (Euler non-incremental) projection method and a modified Euler incremental projection method for non inf-sup stable mixed finite elements are analyzed. The analysis of the classical Euler non-incremental and Euler incremental methods are obtained as a particular case. We first prove that the modified Euler non-incremental scheme has an inherent stabilization that allows the use of non inf-sup stable mixed finite elements without any kind of extra added stabilization. We show that it is also true in the case of the classical Chorin–Temam method. For the second scheme, we study a stabilization that allows the use of equal-order pairs of finite elements. The relation of the methods with the so-called pressure stabilized Petrov Galerkin method (PSPG) is established. The influence of the chosen initial approximations in the computed approximations to the pressure is analyzed. Numerical tests confirm the theoretical results.

1. Introduction

In this paper we analyze a modified Chorin–Temam (Euler non-incremental) projection method for non inf-sup stable mixed finite elements. The analysis of the classical Euler non-incremental method is obtained as a particular case. We prove that both the modified and the standard Euler non-incremental schemes have an inherent stabilization that allows the use of non inf-sup stable mixed finite elements without any kind of extra added stabilization. Although this result is known (see for example [10,18]) to our knowledge there are no proved error bounds for the Chorin–Temam method with non inf-sup stable elements in the literature (see below for related results in [1]). For the closely-related Euler incremental scheme we analyze a modified method for non inf-sup stable pairs of finite elements. In this case an added stabilization is required. The analysis of a stabilized Euler incremental scheme is also obtained as a consequence of the analysis of the modified method. We establish the relation of the methods with the so called pressure stabilized Petrov Galerkin method (PSPG).

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[1] Research supported by Spanish MINECO under grants MTM2013-42538-P (MINECO, ES) and MTM2016-78995-P (AEI/FEDER UE) and by Junta de Castilla y León under grant VA024P17 (FEDER, UE). The author acknowledges the support of European Cooperation in Science and Technology through COST Action IS1104.
[2] Research supported by Spanish MINECO under grant MTM2015-65608-P.
[3] Research supported by Spanish MINECO under grants MTM2013-42538-P (MINECO, ES) and MTM2016-78995-P (AEI/FEDER UE) and by Junta de Castilla y León under grant VA024P17 (FEDER, UE).

https://doi.org/10.1016/j.amc.2017.11.046
0096-3003/

It has been observed in the literature that the standard Euler non-incremental scheme provides computed pressures that behave unstably for Δt small and fixed h if non inf-sup stable elements are used, see [3]. With our error analysis we clarify this question since in that case the inherent PSPG stabilization of the method disappears.

In the present paper, we analyze the influence of the initial approximations to the velocity and pressure in the error bounds for the pressure. In agreement with the results obtained for the PSPG method in [14] a stabilized Stokes approximation of the initial data is suggested as initial approximation. We show both analytically and numerically that with this initial approximation we can obtain accurate approximations for the pressure from the first time step.

Our analysis is valid for any pair of non inf-sup stable mixed finite elements whenever the pressure space Q_h satisfies the condition $Q_h \subset H^1(\Omega)$. However, we prove that the rate of convergence cannot be better than quadratic (in terms of h) for the L^2 errors of the velocity and linear for the L^2 errors of the pressure so that using finite elements other than linear elements in the approximations to the velocity and pressure offers no clear advantage. In terms of Δt the rate of convergence we prove is one for the L^2 errors of the velocity. For the L^2 discrete in time and H^1 in space errors for the velocity and L^2 discrete in time and L^2 in space errors for the pressure the rate of convergence in terms of Δt is one for the modified Chorin–Temam method and is one half for the standard Chorin–Temam method, accordingly to the rate of convergence of the continuous in space Chorin–Temam method, see [11] and the references therein. The analysis presented in this paper is not intended to obtain bounds with constants independent of the viscosity parameter. The possibility of obtaining viscosity independent error bounds will be the subject of further research.

Of course, the Chorin–Temam projection method is well known and this is not the first paper where the analysis of this method is considered. The analysis of the semidiscretization in time is carried out in [16–20]. In [3] the stability of the Chorin–Temam projection method is considered and, in case of non inf-sup stable mixed finite elements, some a priori bounds for the approximations to the velocity and pressure are obtained but no error bounds are proven for this method. In [1] the Chorin–Teman method is considered together with both non inf-sup stable and inf-sup stable mixed finite elements. In case of using non inf-sup stable mixed finite elements a local projection type stabilization is required in [1] to get the error bounds of the method. In the present paper, however, we get optimal error bounds without any extra stabilization for non inf-sup stable mixed finite elements.

For the Euler incremental scheme the analysis of the semidiscretization in time can be found in [16]. The Euler incremental scheme with a spatial discretization based on inf-sup stable mixed finite elements is analyzed in [12]. To our knowledge there is no error analysis for this method in case of using non-inf-sup stable elements. Some stability estimates can be found in [3] for the method with added stabilization terms more related to local projection stabilization than to the PSPG stabilization we consider in the present paper. A stabilized version of the incremental scheme is also proposed in [15] although no error bounds are proved. Finally, for an overview on projection methods we refer the reader to [11].

Being the Chorin–Temam projection method an old one, it has seen the appearance of many alternative methods during the years, many of which possess better convergence properties. The purpose of this paper is not to discuss its advantages or disadvantages with respect to newer methods, but just to analyze its inherent stabilization properties which allow the use of non inf-sup stable elements without extra stabilization, and its connection with (more modern) PSPG stabilization.

For simplicity in the exposition in most of the paper we concentrate on the transient Stokes equations assuming enough regularity for the solution. However, in Section 4.2 we include a sketch of the analysis of the modified Euler non-incremental scheme in the general case in which non-local compatibility conditions for the solution are not assumed. For a detailed analysis including the analysis of the evolutionary Navier–Stokes equations without assuming compatibility conditions we refer the reader to [8].

The outline of the paper is as follows. We first introduce some notation. In the second section we consider the steady Stokes equations and introduce a stabilized Stokes approximation that will be used in the error analysis of the method. Next section is devoted to the analysis of the evolutionary Stokes equations. Both methods Euler non-incremental and Euler-incremental schemes are considered. In the last section some numerical experiments are shown.

2. Preliminaries and notation

Throughout the paper, standard notation is used for Sobolev spaces and corresponding norms. In particular, given a measurable set $\omega \subset \mathbb{R}^d$, $d = 2, 3$, its Lebesgue measure is denoted by $|\omega|$, the inner product in $L^2(\omega)$ or $L^2(\omega)^d$ is denoted by $(\cdot, \cdot)_\omega$ and the notation $(\cdot, \cdot)$ is used instead of $(\cdot, \cdot)_\Omega$. The semi norm in $W^{m,p}(\omega)$ will be denoted by $|\cdot|_{m,p,\omega}$ and, following [7], we define the norm $\|\cdot\|_{m,p,\omega}$ as

$$\|f\|_{m,p,\omega}^p = \sum_{j=0}^{m} |\omega|^{\frac{p(j-m)}{d}} |f|_{j,p,\omega}^p,$$

so that $\|f\|_{m,p,\omega}|\omega|^{\frac{m}{d}-\frac{1}{p}}$ is scale invariant. We will also use the conventions $\|\cdot\|_{m,\omega} = \|\cdot\|_{m,2,\omega}$ and $\|\cdot\|_m = \|\cdot\|_{m,2,\Omega}$. As it is usual we will use the special notation $H^s(\omega)$ to denote $W^{s,2}(\omega)$ and we will denote by $H_0^1(\Omega)$ the subspace of functions of $H^1(\Omega)$ satisfying homogeneous Dirichlet boundary conditions. Finally, $L_0^2(\Omega)$ will denote the subspace of function of $L^2(\omega)$ with zero mean.

Let us denote by $\mathcal{T}_h$ a triangulation of the domain Ω, which, for simplicity, is assumed to be convex with Lipschitz polygonal boundary. On $\mathcal{T}_h$, we consider the finite element spaces $V_h \subset V = H_0^1(\Omega)^d$ and $Q_h \subset L_0^2(\Omega) \cap H^1(\Omega)$ based on