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Short communication

Analytical solution of the flow of a Newtonian fluid with pressure-dependent viscosity in a rectangular duct

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ABSTRACT

We derive a fully analytical solution for the steady flow of an isothermal Newtonian fluid with pressure-dependent viscosity in a rectangular duct. The analytical solution for the governing equations is exact (based on the work by Akyildiz and Siginer, Int. J. Eng. Sc., 104, 2016), while the total mass balance constraint is satisfied with a high-order asymptotic expression in terms of the dimensionless pressure-dependent coefficient ε, and an excellent improved solution derived with Shanks' nonlinear transformation. Numerical calculations confirm the correctness, accuracy and consistency of the asymptotic expression, even for large values of ε. Results for the average pressure difference required to drive the flow are also presented and discussed, revealing the significance of the pressure-dependent viscosity even for steady, unidirectional, Newtonian flow.

1. Introduction and problem definition

During the last few decades, the flow of liquids with pressure-dependent viscosity has become an active area of research by many investigators [1–11]. This effect is important in cases for which a substantial range of the total pressure is developed in the flow domain, or when steep pressure gradients are observed. The nonlinear coupling of the pressure with the velocity field makes flows with pressure-dependent viscosity more difficult to solve than their constant-viscosity counterparts.

In this short note, we study the 2D unidirectional and pressure-driven steady flow of an isothermal Newtonian fluid in a rectangular duct with dimensions $L\times H\times l$. The length of the duct l is assumed much larger than its height H so that the flow is fully developed and entrance/end effects can be neglected. A sketch of the geometry of the flow and the Cartesian coordinate system xyz (used to describe the flow field) with unit vectors $\mathbf{e_x}$, $\mathbf{e_y}$ and $\mathbf{e_z}$ is given in Fig. 1. It is assumed that the mass density ρ^* of the fluid is constant while its viscosity η^* varies linearly with the total pressure p^* [2]:

$$\eta^* = \eta_0^*(1+\beta^*(p^*-p_0^*)) \tag{1}$$

where η_0^* is the viscosity at the reference pressure p_0^*, and β^* is the coefficient of proportionality between the pressure difference and the viscosity (a star denotes dimensional quantity). Typical values for β^* are $10-50\,GPa^{-1}$ for polymer melts [12,13], $10-70\,GPa^{-1}$ for lubricants [14] and $10-20\,GPa^{-1}$ for mineral oils [15].

At steady state, neglecting any external forces and torques, and assuming unidirectional two-dimensional velocity profile $\mathbf{u}^*=w^*(x^*,\ y^*)\mathbf{e_z}$ and three-dimensional pressure p^*, the continuity equation is automatically satisfied, and the three

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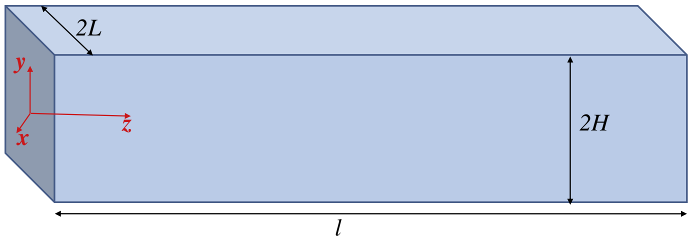

Fig. 1. Sketch of the geometry and coordinate system ($H<<l$, $H<L$). The flow is from left to right.

components of the momentum equation read:

$$-\frac{\partial p^*}{\partial x^*}+\eta_0^*\beta^*\frac{\partial p^*}{\partial z^*}\frac{\partial w^*}{\partial x^*}=0 \tag{2}$$

$$-\frac{\partial p^*}{\partial y^*}+\eta_0^*\beta^*\frac{\partial p^*}{\partial z^*}\frac{\partial w^*}{\partial y^*}=0 \tag{3}$$

$$-\frac{\partial p^*}{\partial z^*}+\eta_0^*\beta^*\left(\frac{\partial p^*}{\partial x^*}\frac{\partial w}{\partial x^*}+\frac{\partial p^*}{\partial y^*}\frac{\partial w^*}{\partial y^*}\right)+\eta_0^*(1+\beta^*(p^*-p_0^*))\left(\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{\partial^2 w^*}{\partial y^{*2}}\right)=0 \tag{4}$$

The equations are rendered dimensionless by scaling x^* by L, y^* by H, z^* by l, η^* by η_0^*, w^* by the average entrance velocity U^*, and $p^*-p_0^*$ by $3\eta_0^*U^*l/H^2$. The scaling for the pressure difference is such that in the case of a constant shear viscosity (i.e. for $\eta^*=\eta_0^*$) the required average pressure drop to drive the flow is unity. With these scales, the dimensionless forms of Eqs. (1–4) become:

$$\eta=1+\varepsilon p \tag{5}$$

$$-3\frac{\partial p}{\partial x}+\varepsilon\, a^2\frac{\partial p}{\partial z}\frac{\partial w}{\partial x}=0 \tag{6}$$

$$-3\frac{\partial p}{\partial y}+\varepsilon\, a^2\frac{\partial p}{\partial z}\frac{\partial w}{\partial y}=0 \tag{7}$$

$$-3\frac{\partial p}{\partial z}+\varepsilon\, c^2\frac{\partial p}{\partial x}\frac{\partial w}{\partial x}+\varepsilon\frac{\partial p}{\partial y}\frac{\partial w}{\partial y}+(1+\varepsilon p)\left(c^2\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{8}$$

Three dimensionless numbers appear. The pressure-dependent coefficient ε and the two aspect ratios, respectively, defined by:

$$\varepsilon\equiv 3\beta^*\eta_0^*U^*l/H^2,\quad a\equiv H/l,\quad c\equiv H/L \tag{9}$$

By setting $H=l$ (i.e. $a=1$) Eqs. (5–8) and ε reduce to those reported by Akyildiz and Siginer [1]. Regarding the magnitude of the dimensionless parameters, ε and a are generally small, while c can take any positive value. Without any loss of generality, it can be assumed, however, that $0<c\le 1$.

Eqs. (5)–(8) are closed with the usual no-slip boundary condition at the walls, i.e. $w(x=\pm1,\ y)=w(x,\ y\pm 1)=0$, and a reference value for the pressure, $p(x=1,\ y=1,\ z=1)=0$. Substituting (6) and (7) into (8) one gets:

$$-3\frac{\partial p}{\partial z}\left(1-\frac{\varepsilon^2c^2a^2}{9}\left(\frac{\partial w}{\partial x}\right)^2-\frac{\varepsilon^2a^2}{9}\left(\frac{\partial w}{\partial y}\right)^2\right)+(1+\varepsilon p)\left(c^2\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{10}$$

The analytical solution of Eq. (10) has been derived recently by Akyildiz & Siginer [1]. First, Eq. (10) is rearranged as follows:

$$\left(c^2\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\Big/\left(1-\frac{\varepsilon^2c^2a^2}{9}\left(\frac{\partial w}{\partial x}\right)^2-\frac{\varepsilon^2a^2}{9}\left(\frac{\partial w}{\partial y}\right)^2\right)=\frac{3\frac{\partial p}{\partial z}}{1+\varepsilon p}=-A \tag{11}$$

where A is a constant that must be determined as part of the solution. The solution of the first equation of (11) is given as:

$$w(x,y)=-\frac{1}{AA_1}\ln\left(u(x,y)+1\right),\quad u(x,y)=\sum_n^\infty v_n(x)G_n(y) \tag{12a}$$