Applied Mathematics and Computation 321 (2018) 689–699

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on "Convergence radius of Osada's method under Hölder continuous condition"

José L. Hueso [a], Eulalia Martínez [a,*], D.K. Gupta [b], Fabricio Cevallos [c]

[a] Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, València, Spain
[b] Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, India
[c] Facultad de Ciencias Económicas, Universidad Laica Eloy Alfaro de Manabí, Manta, Ecuador

ARTICLE INFO

MSC:
65H05
65H10

Keywords:
Nonlinear equations
Iterative methods
Multiple roots
Local convergence
Dynamics

ABSTRACT

In this paper we revise the proofs of the results obtained in "Convergence radius of Osada's method under Hölder continuous condition" [4], because the remainder of the Taylor's expansion used for the obtainment of the local convergence radius is not correct. So we perform the complete study in order to modify the equation for getting the local convergence radius, the uniqueness radius and the error bounds. Moreover a dynamical study for the third order Osada's method is also developed.

1. Introduction

In the last years some of the studies concerning on iterative methods for approximating roots of nonlinear equations have focused on multiple roots. It is a special case where some particular aspects must be taken into account. Some real applications give this problem special interest, see [8], with a study of the multipactor effect, analyzing the trajectory equation of an electron in the air gap between two parallel plates results in a nonlinear equation with a multiple root. This also happens in the Van der Waals equation of state among other phenomenons.

Especially interesting from a mathematical point of view is paper [1] where a complete local convergence study has been performed, obtaining the convergence radius of the well-known modified Newtons method for multiple zeros, when the involved function satisfies a Hölder or a center-Hölder continuity condition. This result is improved in [2]. Similar results for the third order method due to Halley are obtained in [3,9].

We are now interested in this kind of local convergence studies for third order methods for multiple roots. So we center our attention in papers [3] and [4], where the authors analyze the local convergence for Osada and Halleys's method under Hölder and center-Hölder continuity conditions.

We consider the third order method of Osada [4] to find a multiple zero x^* of multiplicity m of a nonlinear equation $f(x) = 0,\ f : D \subseteq: \mathbb{R} \longrightarrow: \mathbb{R},$ given by:

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}. \quad (1)$$

We say that r is the radius of the local convergence ball if the sequence x_n generated by this iterative method, starting from any initial point in the open ball $B(x^*, r)$ converges to x^* and remains in the ball. In these studies it is interesting to obtain

* Corresponding author.
E-mail address: eumarti@mat.upv.es (E. Martínez).

https://doi.org/10.1016/j.amc.2017.11.003

the largest possible value of r, but obviously, this depends on the conditions that the nonlinear function verifies. Here we consider that f satisfies the following Hölder continuous conditions $\forall x, y \in D$,

$$|f^{(m)}(x^*)^{-1}(f^{(m+1)}(x) - f^{(m+1)}(y))| \leq K_0|x-y|^p, \quad K_0 > 0, p \in]0,1]. \tag{2}$$

$$|f^{(m)}(x^*)^{-1}f^{(m+1)}(x)| \leq K_m, \quad \forall x \in D, \quad K_m > 0. \tag{3}$$

Unfortunately, the Taylor's expansion used by the authors of [4] in the proof of Lemma 1 is not correct. The same authors use the correct version of the remainder in Taylor's expansion in the paper "On the convergence radius of the modified Newton method for multiple roots under the center-Hölder condition", see Lemma 1 of [2].

In [4], the authors consider the following formula for Taylor's expansion with integral remainder:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}(x-a)^2 f''(a) + \frac{1}{3!}(x-a)^3 f'''(a) + \cdots$$
$$+ \frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{1}{n!}\int_a^x (f^{(n+1)}(t) - f^{(n+1)}(a))(x-t)^n dt.$$

It is well know that for a Taylor expansion of order n, the derivative evaluated in the remainder is of order $n+1$, but if one uses the integral form remainder, this derivative is of order n. That is, the Taylor's expansion with integral form remainder [7] has the form

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + E_n(x),$$

where

$$E_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

Another way to express the error is

$$E_n(x) = \frac{1}{(n-1)!}\int_a^x (f^{(n)}(t) - f^{(n)}(a))(x-t)^{n-1}dt,$$

where one can check the last equality by writing the last integral as $\int_\alpha^x u dv$ with $u = f^{(n)}(t) - f^n(\alpha)$ and $dv = (x-t)^{n-1}dt$.

In order to correct the results obtained in paper [4], we use different results involving divided differences that are introduced in the following section.

2. Preliminaries

We recall the definitions of divided differences and their properties.

Definition 2.1 [5] The divided differences $f[a_0, a_1, \ldots, a_k]$, on $k+1$ different points $a_0, a_1, \ldots, a_k$ of a function $f(x)$ are defined by

$$f[a_0] = f(a_0),$$
$$f[a_0, a_1] = \frac{f[a_0] - f[a_1]}{a_0 - a_1},$$
$$\vdots$$
$$f[a_0, a_1, \ldots, a_k] = \frac{f[a_0, a_1, \ldots, a_{k-1}] - f[a_1, a_2, \ldots, a_k]}{a_0 - a_k}.$$

If the function f is sufficiently differentiable, then its divided differences $f[a_0, a_1, \ldots, a_k]$ can be defined if some of the arguments a_i coincide. For instance, if $f(x)$ has kth derivative at a_0, then it makes sense to define

$$f\Big[\underbrace{a_0, a_0, \ldots, a_0}_{k+1}\Big] = \frac{f^{(k)}(a_0)}{k!}. \tag{4}$$

Lemma 1. *[5] The divided differences $f[a_0, a_1, \ldots, a_k]$ are symmetric functions of their arguments, i.e., they are invariant under permutations of $[a_0, a_1, \ldots, a_k]$.*

Lemma 2 ([6]). *If the function f has kth derivative, and $f^{(k)}(x)$ is continuous in the interval $I_x = [min(x_0, x_1, \ldots, x_k), max(x_0, x_1, \ldots, x_k)]$ then*

$$f[x_0, x_1, \ldots, x_k] = \int_0^1 \cdots \int_0^1 t_1^{k-1} t_2^{k-2} \ldots t_{k-1} f^{(k)}(t) dt_1 \ldots dt_k,$$