Applied Mathematics and Computation 321 (2018) 1–10

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Polyhedral graphs via their automorphism groups

Modjtaba Ghorbani*, Mahin Songhori

*Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, 16785-136, Islamic Republic of Iran*

ARTICLE INFO

*MSC:*
20D15

*Keywords:*
Polyhedral graph
Cayley graph
Automorphism group
Sylow *p*-subgroup

ABSTRACT

A polyhedral graph is a three connected simple planar graph. An automorphism of a graph is a bijection on its vertices which preserves the edge set. In this paper, we compute the automorphism group of cubic polyhedral graphs whose faces are triangles, quadrangles, pentagons and hexagons. In continuing, we classify all cubic polyhedral graphs with Cayley graph structure.



## 1. Introduction

All considered graphs in this paper are simple and connected without loops and parallel edges. A classical fullerene is a cubic three connected graph whose faces entirely composed of pentagons and hexagones and we denote it by a PH-fullerene or briefly by a fullerene, see [20,21]. The non-classical fullerenes are composed of triangles and hexagones or squares and hexagones and we denote them by TH-fullerene or SH-fullerene, respectively. There are many problems concerning with fullerene graphs and many properties of them are derived, see [2,3,5,9–11] as well as [13,15,18,19,22]. Fullerenes are special cases of a larger class of graphs, namely polyhedral graphs. A polyhedral graph is a three connected simple planar graph and in this paper, we consider only the cubic polyhedral graphs whose faces are a combination of triangles, squares, pentagons and hexagones, see [5,7]. A Cubic polyhedral graph with $t$ triangular, $s$ quadrilateral, $p$ pentagonal, $h$ hexagonal faces and no other faces is denoted by a $(t, s, p, h)$-polyhedral or briefly a $(t, s, p)$-polyhedral graph.

An automorphism of graph $\Gamma = (V, E)$ is a bijection $\beta$ on $V$ which preserves the edge set $E$. In other words, $e = uv$ is an edge of $E$ if and only if $e^\beta = u^\beta v^\beta$ is an edge of $E$. Here, the image of vertex $u$ is denoted by $u^\beta$. The set of all automorphisms of graph $\Gamma$ with the operation of composition is a group on $V(\Gamma)$ denoted by $Aut(\Gamma)$. Frucht [14] was the first mathematician who dealt with graph automorphism. Also quantitative measures based on graph automorphism have been developed in [4,24].

By above notation, a SPH-polyhedral graph is a planar graph whose faces are squares, pentagons and hexagons. Let $m$ be the number of edges in a given SPH-polyhedral graph $F$. Since each atom lies in exactly three faces and each edge lies in two faces, the number of atoms is $n = (4s + 5p + 6h)/3$, the number of edges is $m = (4s + 5p + 6h)/2 = 3n/2$ and the number of faces is $f = s + p + h$. By using Euler's formula, we have $n - m + f = 2$ and then we deduce that $(4s + 5p + 6h)/3 - (4s + 5p + 6h)/2 + s + h + p = 2$. Hence $p = 12 - 2s$ and $h = n/2 + s - 10$. But $p \geq 0$ yields $s \leq 6$ and the following cases hold:

*Case 1*. $s = 0$ and hence $F$ is a classical fullerene the order of its automorphism group divides 120. They described in [23].

*Cases 2–6*. $s = 1, 2, 3, 4$ or 5 and so $p = 10, 8, 6, 4$ or 2, respectively.

*Case 7*. $s = 6$, then $p = 0$ and $F$ is a fullerene the order of its automorphism group divides 48. These graphs studied in [17].

* Corresponding author.
*E-mail addresses:* mghorbani@srttu.edu, ghorbani30@gmail.com (M. Ghorbani).

https://doi.org/10.1016/j.amc.2017.10.028
0096-3003/

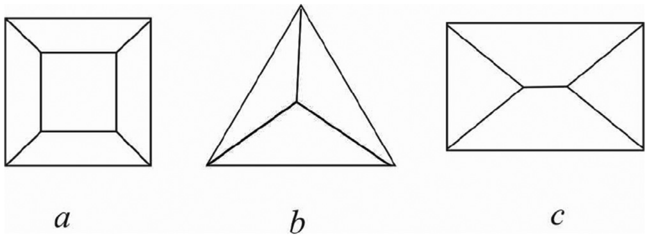


**Fig. 1.** A cube ($a$), the smallest TPH-polyhedral graph ($b$) and the smallest TSH-polyhedral graph ($c$).

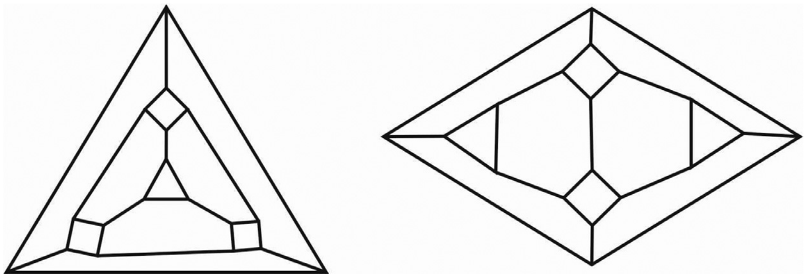

**Fig. 2.** Two members of TSH polyhedral graphs.

A TPH-polyhedral graph $F$ is one whose faces are triangles, pentagons and hexagons. Let $t$ be the number of triangles in $F$. Similar to the last case, we can verify that $p = 12 - 3s$ and $h = n/2 + s - 10$. Since $p \geq 0$, we have $s \leq 4$ and the following cases hold:

Case 1 $s = 0$, then $F$ is a classical fullerene graph whose automorphism group is studied in [23].
Cases 2–4 $s = 1,\ 2,\ 3$ and so $p = 9,\ 6,\ 3$, respectively.
Case 5 $s = 4$, then $p = 0$ and $F$ is a TH-fullerene graph whose automorphism group is studied in [16]. The order of automorphism group of $F$ divides 24.

The smallest *SPH*-polyhedral graph is the cube $Q_3$, a graph on eight vertices without any hexagonal faces, see Fig. 1($a$). The symmetry group of this graph is isomorphic with $\mathbb{Z}_2 \times \mathbb{S}_4$, see [8]. The smallest TPH-polyhedral graph is the pyramid graph depicted in Fig. 1($b$). It is composed of four triangles without pentagons and hexagons. Finally, in a TSH-polyhedral graph with $t$ triangles, $s$ squares and $h$ hexagons, we have $n = (3t + 4s + 6h)/3$, $m = (3t + 4s + 6h)/2$ and the number of faces is $f = t + s + h$. By Euler's formula $3t = 12 - 2s$ and so $s \leq 6$. Since 3 divides $12 - 2s$, we have $s = 0$ or 3 or 6. If $s = 0$, then $F$ is a TH-fullerene. If $s = 6$, then $t = 0$ and $F$ is a SH-fullerene. If $s = 3$, then $t = 2$ and $F$ is a polyhedral graph with exactly two triangles, three squares and $h$ hexagons, see Fig. 2($c$).

The aim of this paper is to study the order of automorphism group of introduced polyhedral graphs.

## 2. Definitions and preliminaries

Let $G$ be a group and $\Omega$ a non-empty set. An action of $G$ on $\Omega$ denoted by $(G|\Omega)$ induces a group homomorphism $\varphi$ from $G$ into the symmetric group $S_\Omega$ on $\Omega$, where $\phi(g)(\alpha) = \alpha^g$ $(\alpha \in \Omega)$. The orbit of an element $\alpha \in \Omega$ is denoted by $\alpha^G$ and it is defined as the set of all $\alpha^g$ where $g \in G$. Size of $\Omega$ is called the degree of this action. The kernel of this action is denoted by $Ker\varphi$. An action is faithful if $Ker\varphi = \{1\}$. The stabilizer of element $\alpha \in \Omega$ is defined as $G_\alpha = \{g \in G | \alpha^g = \alpha\}$. Let $H = G_\alpha$, then for $\alpha,\ \beta \in \Omega\,(\alpha \neq \beta)$, $H_\beta$ is denoted by $G_{\alpha,\ \beta}$. The orbit-stabilizer theorem implies that $|\alpha^G|.|G_\alpha| = |G|$. For every $g \in G$, let $fix(g) = \{\alpha \in X, \alpha^g = \alpha\}$, then we have:

**Lemma 1** (Cauchy–Frobenius Lemma)**.** *Let G acts on set* $\Omega$, *then the number of orbits of G is*

$$\frac{1}{|G|}\sum_{g \in G} |fix(g)|.$$

**Example 1.** Consider the fullerene graph $F_{96}$ depicted in Fig. 3. If $\alpha$ denotes the rotation of $F_{96}$ through an angle of 60° around an axis through the midpoints of the front and back faces, then the corresponding permutation is $\alpha = (1,2,3,4,5,6)(7,10,14,17,20,24)(8,11,15,18,21,25)\ (9,12,16,19,23,26)\ (13,50,58,74,66,42)\ (22,71,47,39,55,63)$ $(27,28,29,30,31,32)(33,48,56,72,64,40)(34,49,57,73,65,41)(35,51,59,75,67,43)(36,52,60,76,68,44)(37,53,61,77,$ $69,45)(38,54,62,78,70,46)\ \ (79,80,81,82,83,84)(85,86,87,88,89,90)(91,96,95,94,93,92)$. Thus, one of orbits of subgroup $\langle\alpha\rangle$ containing the vertex 1 is $1^{\langle\alpha\rangle} = \{1,2,3,4,5,6\}$. Now, consider the axis symmetry element which fixes vertices {1, 4, 8, 18, 43, 44, 59, 60, 85, 88, 92, 95}, the corresponding permutation is $\beta = (2,6)\ (3,5)\ (7,9)\ (10,26)$