Applied Mathematics and Computation 321 (2018) 25–36

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# The increase in the resolvent energy of a graph due to the addition of a new edge

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ARTICLE INFO

*Keywords:*
Resolvent energy
Resolvent energy matrix
Characteristic polynomial

ABSTRACT

The resolvent energy $ER(G)$ of a graph $G$ on $n$ vertices whose adjacency matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$ is the sum of the reciprocals of the numbers $n-\lambda_1, \ldots, n-\lambda_n$. We introduce the resolvent energy matrix $\mathbf{R}(G)$ and present an algorithm that produces this matrix. This algorithm may also be used to update $\mathbf{R}(G)$ when new edges are introduced to $G$. Using the resolvent energy matrix $\mathbf{R}(G)$, we determine the increase in the resolvent energy $ER(G)$ of $G$ caused by such edge additions made to $G$. Moreover, we express this increase in terms of the characteristic polynomial of $G$ and the characteristic polynomials of three vertex-deleted subgraphs of $G$.



## 1. Introduction

Let $G$ be a simple graph on $n$ vertices having vertex set $\mathcal{V}(G)=\{1,2,3,\ldots,n\}$ and edge set $\mathcal{E}(G)$. Two vertices $u$ and $v$ are adjacent in $G$ if and only if $\{u,v\}\in\mathcal{E}(G)$. If $\{u,v\}\notin\mathcal{E}(G)$, then the graph $G+uv$ is the graph with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)\cup\{\{u,v\}\}$. If $H$ has the same number of vertices as $G$, then $G$ is a proper subgraph of $H$ if $\mathcal{E}(G)\subset\mathcal{E}(H)$. The graph $G-u$ denotes the one-vertex-deleted subgraph of $G$ obtained from $G$ after removing vertex $u$ and the edges incident to $u$. The graph $G-u-v$ denotes the two-vertex-deleted subgraph $(G-u)-v$ of $G$.

Let $\mathbf{A}$ be the $n\times n$ adjacency matrix of $G$. The graph $G$ has characteristic polynomial $\phi(G,x)=\det(x\mathbf{I}-\mathbf{A})$, where $\mathbf{I}$ is the identity matrix. The roots of $\phi(G,x)$ are the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ of $\mathbf{A}$. The complete graph $K_n$ on $n$ vertices is the graph whose $n\times n$ adjacency matrix is $\mathbf{J}-\mathbf{I}$, where $\mathbf{J}$ is the matrix of all ones. On the other hand, the empty graph $N_n$ on $n$ vertices is the graph whose adjacency matrix is the $n\times n$ zero matrix.

A walk of length $\ell$ in $G$ is a sequence of vertices $v_0, v_1, \ldots, v_\ell$ of $G$ such that $\{v_i, v_{i+1}\}\in\mathcal{E}(G)$ for all $i\in\{0,1,2,\ldots,\ell-1\}$. Such a walk is closed if $v_0=v_\ell$. The $k$th spectral moment $M_k(G)$ of $G$ is the sum of the $k$th powers of all of the eigenvalues of its adjacency matrix. Since $\mathrm{tr}(\mathbf{M})$, the trace of a matrix $\mathbf{M}$, is equal to the sum of the eigenvalues of $\mathbf{M}$ [21], $M_k(G)=\mathrm{tr}(\mathbf{A}^k)$. Moreover, it is well known that the entry in the $j$th row and $k$th column of $\mathbf{A}^\ell$ is equal to the number of walks of length $\ell$ in $G$, starting from $j\in\mathcal{V}(G)$ and ending at $k\in\mathcal{V}(G)$ [7]. Thus, $M_k(G)$ may be thought of as being the total number of closed walks of length $k$ in $G$, starting and ending at any vertex.

In 1978, Ivan Gutman, motivated by research on the total $\pi$-electron energy of molecules, defined the *graph energy* [15] as $\sum_{i=1}^n|\lambda_i|$. Starting from 2006, a surprisingly high number of graph energy variants were proposed in the literature, each with their own applications. This 'energy deluge' is discussed in reference [16], which additionally surveys and compares several of these graph energy variants. For a more thorough discussion of many such alternative graph energies, the reader

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https://doi.org/10.1016/j.amc.2017.10.020
0096-3003/

is referred to the books [20,22]. Moreover, in a recent paper [24], new upper bounds were produced for several of these graph energies.

One of the more recent of these graph energy variants, the *resolvent energy*, was introduced in [19], following the earlier works by Estrada and Higham [12], and Chen and Qian [5]. It is defined by

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}.$$

Eventually, the resolvent energy was extensively studied [1,9,14,17,18]. Also, its Laplacian spectrum version was recently put forward [3,25].

In [19, Theorem 2], it was shown that

$$ER(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{n^{k+1}}.$$

Thus, the resolvent energy belongs to a general class of cumulative vertex centrality measures based on closed walks, originally put forward by Estrada and Higham in [12]. This class contains graph invariants of the form

$$\mathbb{E}(G) = \sum_{k=0}^{\infty} c_k M_k(G) \tag{1}$$

with the sequence of positive real numbers $c_0, c_1, c_2, \ldots$ chosen such that the Maclaurin series $\sum_{k=0}^{\infty} c_k x^k$ converges to some function $f(x)$. Since $M_k(G) = \operatorname{tr}(\mathbf{A}^k)$, we have the relation

$$\mathbb{E}(G) = \operatorname{tr}\big(f(\mathbf{A})\big).$$

For instance, when $-n < x < n$, the series $\sum_{k=0}^{\infty} n^{-k-1} x^k$ converges to $(n-x)^{-1}$. Since the eigenvalues of $\mathbf{A}$ also satisfy this inequality for any graph $G$ (see, for example, [26]), the summation $\sum_{k=0}^{\infty} n^{-k-1}\mathbf{A}^k$ converges to $(n\mathbf{I}-\mathbf{A})^{-1}$. Note that the eigenvalues of $(n\mathbf{I}-\mathbf{A})^{-1}$ are $\frac{1}{n-\lambda_1}, \ldots, \frac{1}{n-\lambda_n}$, all of which are positive real numbers; hence, this inverse matrix exists for all graphs and is positive-definite. The resolvent energy $ER(G)$ is thus $\mathbb{E}(G)$ with $c_k = \dfrac{1}{n^{k+1}}$ for all $k$ and with $f(x) = \dfrac{1}{n-x}$. The following lemma is consequently inferred.

**Lemma 1.1.** $ER(G) = \operatorname{tr}\big((n\mathbf{I}-\mathbf{A})^{-1}\big)$.

Two other particular cases of graph invariants pertaining to the class $\mathbb{E}(G)$ of the form (1) are the *Estrada index* [4,8,10,11,13], in which

$$c_k = \frac{1}{k!} \text{ for all } k,\ f(x) = e^x,\ \mathbb{E}(G) = EE(G) = \operatorname{tr}\big(e^{\mathbf{A}}\big)$$

and the *resolvent Estrada index* [2,5,12] (defined for graphs that are not complete) in which

$$c_k = \frac{1}{(n-1)^k} \text{ for all k},\ \ f(x) = \frac{n-1}{n-1-x},$$
$$\mathbb{E}(G) = EE_r(G) = (n-1)\operatorname{tr}\big(((n-1)\mathbf{I}-\mathbf{A})^{-1}\big).$$

Clearly, there is a relation between the resolvent Estrada index $EE_r(G)$ and the resolvent energy $ER(G)$. Indeed, they are both based on the *resolvent matrix* of $\mathbf{A}$, defined by $(z\mathbf{I}-\mathbf{A})^{-1}$, where $z$ is a complex variable [28]. The resolvent matrix of $\mathbf{A}$ exists for values of $z$ that are not eigenvalues of $\mathbf{A}$.

It is clear, by Lemma 1.1, that studying the matrix $(n\mathbf{I}-\mathbf{A})^{-1}$ should elucidate research on the resolvent energy. Because of this, we first establish strict bounds for the entries of the matrix $(n\mathbf{I}-\mathbf{A})^{-1}$ in Section 2. Subsequently, we consider how this matrix changes after introducing a new edge to a graph $G$, leading to the algorithm in Section 4 that evaluates the resolvent energy of any graph without the need of evaluating any matrix inverse or any eigenvalues. In Section 5, the resolvent energy change $\delta$ caused by the introduction of a new edge in $G$ is quantified using entries of $(n\mathbf{I}-\mathbf{A})^{-1}$. After deriving expressions for the entries of this matrix in terms of four characteristic polynomials related to $G$, we present a formula in Section 7 that evaluates $\delta$ from these characteristic polynomials.

## 2. The resolvent energy matrix

Motivated by the previous introductory section, we start this section by making the following definition.

**Definition 2.1.** The *resolvent energy matrix* of a graph $G$ on $n$ vertices having adjacency matrix $\mathbf{A}$ is the matrix $\mathbf{R}(G) = (n\mathbf{I}-\mathbf{A})^{-1}$.

We denote the resolvent energy matrix $\mathbf{R}(G)$ of Definition 2.1 by $\mathbf{R}$ if the graph $G$ is clear from the context. Note that $\mathbf{R}$ has rational entries, since it is the inverse of a matrix with integer entries. Because of this, $ER(G) = \operatorname{tr}(\mathbf{R}(G))$ is a rational