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The eigenvalues range of a class of matrices and some applications in Cauchy–Schwarz inequality and iterative methods☆

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ABSTRACT

This paper discusses the range of the eigenvalues of a class of matrices. By using the eigenvalues range of a class of matrices, an extension of the inner product type Cauchy–Schwarz inequality is obtained, the convergence proof of the least squares based iterative algorithm for solving the coupled Sylvester matrix equations is given and the best convergence factor is determined. Moreover, by using the eigenvalues range of this class of matrices, an iterative algorithm for solving linear matrix equation is established. Three numerical examples are offered to illustrate the effectiveness of the results suggested in this paper.

1. Introduction

Matrix eigenvalue theory plays an important role in areas such as the convergence analysis of matrix equation iterative algorithms [1–3] and the stability analysis of control theory [4–6]. For example, by using the maximum eigenvalue of the related matrices, the convergence analysis of the gradient-based iterative algorithm for some matrix equations were suggested [7,8] and the range of convergence factor of the gradient-based iterative algorithm for a class of matrix equations was determined [9–11].

Much work has been done on the eigenvalues of matrix [12,13] and some matrix eigenvalues inequalities were established [14,15]. For example, a new proof method for the arithmetic and geometric mean inequality of the singular values for any two matrices was given in [16]. To prove the convergence of the least squares based iterative algorithm of the coupled Sylvester matrix equations, the structure and range of the eigenvalues related to the symmetric positive definite matrix were discussed [17].

For its important applications in the stability analysis of control theory, the matrix equation is an active research topic. Some techniques from the system identification of the control theory [18–21] have been adapted to the matrix equation field. For example, by introducing the hierarchical identification principle [22], some coupled or extended Sylvester matrix equations were discussed in [23,24]. Finite iterative algorithm is another popular method for solving matrix equations, by using the conjugate gradient squared method, some generalized coupled Sylvester matrix equations were discussed in [25,26].

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By using the hierarchical identification principle, the least squares based iterative algorithm for solving the coupled Sylvester matrix equations was established [27]. But only the sufficient condition of the convergence factor for the convergence of the iterative algorithm was determined and the best choice of the convergence factor was not determined.

Inspiring by the above work and especially to determine the necessary and the sufficient conditions and the best choice of the convergence factor of the least squares based iterative algorithm for the coupled Sylvester matrix equations, in this paper, the eigenvalues of a class of matrices are discussed and the range of these eigenvalues is determined. By introducing the principal angles of two different subspaces, an extension of the inner product type Cauchy–Schwarz inequality is established. The convergence proof of the least squares based iterative algorithm for solving the coupled Sylvsester matrix equations is offered and the best choice of the convergence factor is determined. Moreover, an iterative algorithm for solving linear matrix equation is presented.

The rest of this paper is organized as follows. Section 2 gives some notation and a preliminary lemma. Section 3 discusses and determines the eigenvalues range of a class of matrices. Section 4 establishes an extension of the inner product type Cauchy–Schwarz inequality. Using the result established in Section 3, Section 5 determines the range and structure of a class of matrices related to the symmetric positive definite matrix. Section 6 proves the convergence of the least squares based iterative for solving the coupled Sylvester matrix equations. Section 7 establishes an iterative algorithm for solving matrix equation $\boldsymbol{AX} = \boldsymbol{F}$. Section 8 offers three numerical examples to illustrate the effectiveness of the suggested algorithms. Finally, we offer some concluding remarks in Section 9.

2. Basic preliminaries

Let us introduce some notation and a lemma first. $\boldsymbol{I}_n$ is the identity matrix with order $n \times n$. $\boldsymbol{0}$ denotes a zero matrix with proper orders. For a square matrix $\boldsymbol{A}$, we use symbols $\lambda[\boldsymbol{A}]$, $\rho(\boldsymbol{A})$, $\det(\boldsymbol{A})$, $\boldsymbol{A}^{\mathrm{T}}$, $\boldsymbol{A}^{-1}$, $\mathrm{tr}(\boldsymbol{A})$ and $\mathrm{rank}[\boldsymbol{A}]$ to represent respectively the set of the eigenvalues, the spectral radius, the determinant, the transpose, the inverse, the trace and the rank of $\boldsymbol{A}$. $\|\boldsymbol{\alpha}\|$ denotes the vector norm and is defined by formula $\|\boldsymbol{\alpha}\|^2 := \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}$. $\|\boldsymbol{A}\|$ represents the Frobenius norm of matrix $\boldsymbol{A}$ and defined by formula $\|\boldsymbol{A}\|^2 := \mathrm{tr}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$. $\mathcal{F}$ denotes the subspace spanned by the row vectors of matrix $\boldsymbol{F}$. $\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denotes a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. $\max\{\mathcal{S}\}$ ($\min\{\mathcal{S}\}$) denotes the maximum (minimum) element of the set $\mathcal{S}$.

An idempotent matrix has the following orthogonal decomposition.

Lemma 1. *If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is a row-full rank matrix, then matrix $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{A}$ is idempotent and all the eigenvalues of matrix $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{A}$ are 1 or 0, that is, there exists an orthogonal matrix $\boldsymbol{Q}$ such that*

$$\boldsymbol{Q}^{\mathrm{T}}[\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{A}]\boldsymbol{Q} = \mathrm{diag}\{1, \ldots, 1, 0, \ldots, 0\} =: \boldsymbol{\Lambda}.$$

Furthermore, we have $\mathrm{rank}[\boldsymbol{\Lambda}] = m$.

This lemma appeared in [17], for the proof one can refer to [17].

The principal angle of two subspaces is an extension of the angle formed by two vectors [12,13]. Let $\mathcal{F}$ and $\mathcal{G}$ be subspaces in $\mathbb{R}^n$ whose dimensions satisfy

$$p = \dim(\mathcal{F}) \geqslant \dim(\mathcal{G}) = q \geqslant 1.$$

If $\dim(\mathcal{F} \cap \mathcal{G}) = 0$ then the nonzero principal angles $\theta_1, \theta_2, \ldots, \theta_q \in [0, \pi/2]$ between $\mathcal{F}$ and $\mathcal{G}$ are defined recursively by

$$\cos\theta_k = \max_{\boldsymbol{u}\in\mathcal{F}} \max_{\boldsymbol{v}\in\mathcal{G}} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}_k^{\mathrm{T}}\boldsymbol{v}_k, \tag{1}$$

where the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy

$$\begin{aligned} \|\boldsymbol{u}\| &= \|\boldsymbol{v}\| = 1, \\ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}_i &= 0, \ \ i = 1 : k-1, \\ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}_i &= 0, \ \ i = 1 : k-1. \end{aligned}$$

Here, the principal angles satisfy $0 \le \theta_1 \le \theta_2 \cdots \le \theta_q \le \pi/2$. Referring to [28], $\sin\theta(\mathcal{F}, \mathcal{G})$ can be defined as $\sin\theta(\mathcal{F}, \mathcal{G}) := \sin\theta_1 \sin\theta_2 \ldots \sin\theta_q$ and $\cos\theta(\mathcal{F}, \mathcal{G})$ can be defined as $\cos\theta(\mathcal{F}, \mathcal{G}) := \cos\theta_1 \cos\theta_2 \ldots \cos\theta_q$. In particular, $\sin^2\theta(\mathcal{F}, \mathcal{G}) + \cos^2\theta(\mathcal{F}, \mathcal{G}) \leqslant 1$.

3. Range of the eigenvalues of a class of matrices

In this section, we discuss the range of the eigenvalues of a class of matrices. Let $\boldsymbol{F} \in \mathbb{R}^{m\times t}$ and $\boldsymbol{G} \in \mathbb{R}^{n\times t}$ be two row-full rank matrices with $m \le t$, $n \le t$ and $m \ge n$. Set

$$\boldsymbol{P} := (\boldsymbol{G}\boldsymbol{G}^{\mathrm{T}})^{-1}\boldsymbol{G}[\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}})^{-1}\boldsymbol{F}]\boldsymbol{G}^{\mathrm{T}} \in \mathbb{R}^{n\times n}. \tag{2}$$

Let ρ be the eigenvalue of $\boldsymbol{P}$. we have the following result.

Theorem 1. *Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two row-full rank matrices and $\boldsymbol{P}$ is defined in Eq. (2). If ρ is the eigenvalue of the matrix $\boldsymbol{P}$ then $0 \le \rho \le 1$.*