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Fractional-order Legendre-collocation method for solving fractional initial value problems

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ABSTRACT

In this paper, we present a numerical algorithm for solving second-order fractional initial value problems. This numerical algorithm is based on a fractional Legendre-collocation spectral method. The governing fractional differential equation is converted into a nonlinear system of algebraic equations. The error analysis of the proposed numerical algorithm is presented. Comparisons with other numerical methods shows that the proposed algorithm is more accurate and simpler to implement. Several examples are discussed to illustrate the efficiency and accuracy of the present scheme.

1. Introduction

In recent years, fractional calculus (the branch of calculus that generalizes the derivative of a function to non-integer order) has been a subject of numerous investigations by scientists from mathematics, physics and engineering communities. The interest in this area of research arises mainly from its applications to many models in the fields of fluid mechanics, electromagnetic, acoustics, chemistry, biology, physics and material sciences; see, by way of example not exhaustive enumeration, [1–12].

In this study, we develop an efficient numerical algorithm for solving a class of fractional initial value problems of the form:

$$D^{\beta}y(x) + f(x, y, y') = 0 \quad x \in (0, 1], \quad 1 < \beta \le 2, \tag{1}$$

subject to

$$y(0) = h_0, \quad y'(0) = h_1, \tag{2}$$

where $h_0, h_1 \in \mathbb{R}$ and $y \in L_1(a, b)$. The notation D^{β} for any $\beta \in \mathbb{R}^+$ denotes the left-sided Caputo fractional derivative. The literature reveals several studies on certain classes of fractional initial-value problems (FIVPs) of the form:

$$F(x, D^{\delta}y(x), D^{\nu_1}y(x), \ldots, D^{\nu_n}y(x)) = 0, \quad y^{(i)}(0) = y_i, \quad 0 \le i \le n, \tag{3}$$

where $n = \lceil \beta \rceil$ is the smallest integer greater than or equal to δ, and $0 < \nu_1 < \nu_2 < \cdots < \nu_n \le \delta$. Therefore, several numerical algorithms are proposed to solve this problem. for instance, Kazem et al. [13] applied fractional-order Legendre Spectral Galerkin method, Bhrawy and Zaky [14] implemented the shifted fractional-order Jacobi orthogonal functions, and Rehman and Khan [15] implemented the Legendre wavelet method. It should be noted that the fractional-Legendre functions have

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been implemented in the so-called fractional-Legendre–Galerkin spectral method by few researchers in order to solve several types of fractional ordinary differential equations, see for example Kazem et al. [13], Klimek and Agrawal [16], Bhrawy and Alghamdi [17], Yiming et al. [18], Syam et al. [19], Syam and Al-Refai [20], Kashkari and Syam [21], and Bhrawy et al. [22].

The present work is motivated by the desire to find an approximate solution of problems (1) and (2) using an efficient numerical technique based on the fractional-Legendre-collocation spectral method. In addition, the error analysis of the proposed numerical algorithm is presented. It is worth mentioning that the present technique can be easily extended to handle problem (3).

The rest of the paper is organized as follows. A brief review of the fractional calculus is presented in Section 2. The proposed method and error analysis results are discussed in Section 3. The numerical results are presented in Section 4.

2. Preliminary results

In this section, we present some basic definitions and properties of fractional calculus theory and the Fractional-order Legendre functions.

2.1. Fractional calculus

In this section, we present some essential preliminaries related to fractional calculus theory.

Definition 1. The left-sided Riemann–Liouville fractional integral operator of order α is defined by

$$J^{\alpha}_{0^+}y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}y(t)dt, \quad \alpha \in \mathbb{R}^+, \tag{4}$$

where, $x \in [0, T]$, y belongs to the Lebesgue space $L_1[0, T]$ and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt.$$

Lemma 1. *([23, 24, 25]) Let $\alpha, \beta > 0$, $x \in [0, T]$, $\gamma > -1$ and $y \in L_1[0, T]$. Then*

(i) $J^0_{0^+}y(x) = y(x)$,

(ii) $J^{\alpha}_{0^+}J^{\beta}_{0^+}y(x) = J^{\alpha+\beta}_{0^+}y(x) = J^{\beta}_{0^+}J^{\alpha}_{0^+}\ y(x)$,

(iii) $J^{\alpha}_{0^+}x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}x^{\gamma+\alpha}$.

Definition 2. For $\alpha \in \mathbb{R}^+$, $m = \lceil\alpha\rceil$ and $x \in [0, T]$, the left-sided Caputo fractional derivatives operator is defined as:

$$D^{\alpha}_{0^+}y(x) = J^{m-\alpha}_{0^+}y^{(m)}(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}y^{(m)}(t)dt, \tag{5}$$

provided the integral exists. This operator was introduced by the Italian mathematician Caputo in 1967, see [26].

Lemma 2. *For $\alpha \in \mathbb{R}^+$, $m = \lceil\alpha\rceil$, $x \in [0, T]$ and $y \in L_1[0, T]$, we have*

1. $D^{\alpha}_{0^+}\ J^{\alpha}_{0^+}y(x) = y(x)$.
2. $J^{\alpha}_{0^+}\ D^{\alpha}_{0^+}y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\dfrac{x^k}{k!}$.
3. $D^{\alpha}_{0^+}x^r = \begin{cases} \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}x^{r-\alpha}, & \text{for } r > \lfloor\alpha\rfloor, \\ 0, & \text{if } r \in \mathbb{N}_0 \ \&\ r \le \lfloor\alpha\rfloor, \\ \text{Does not exists}, & \text{if } r \notin \mathbb{N}_0 \ \&\ r < \lfloor\alpha\rfloor, \end{cases}$

where $\lfloor . \rfloor$ and $\lceil . \rceil$ are the floor and ceiling functions, respectively.

2.2. Fractional-order Legendre functions

The analytical closed form of the shifted Legendre polynomials of degree n is given by

$$L_n(t) = \sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(n-k)!(k!)^2}t^k, \quad t \in (0, 1). \tag{6}$$

It is well-known that the set of polynomial functions $\{L_0, L_1, \ldots\}$ is orthogonal on [0, 1] with respect to the weight function $w(t) = 1$; i.e.