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A triangular spectral element method for elliptic and Stokes problems[☆]

Jingliang Li[a,b], Heping Ma[a,*], Huiyuan Li[c]

[a] Department of Mathematics, College of Sciences, Shanghai University, Shanghai 200444, China
[b] Jiangsu University of Science and Technology, Zhangjiagang, Jiangsu 215600, China
[c] State Key Laboratory of Computer Science Laboratory of Parallel Computing, Institute of Software Chinese Academy of Sciences, Beijing 100190, China

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ABSTRACT

In this paper, we study a triangular spectral-element method based on a one-to-one mapping between the rectangle and the triangle. We construct a new approximation space where the integral singularity brought by the mapping can be removed in a naive and stable way. We build aquasi-interpolation triangular spectral-element approximation, and analyze its approximation error. Based on this quasi-interpolation spectral-element approximation, we put forward a new triangular spectral-element method for the elliptic problems. We present the approximation scheme, analyze the convergence, and do some experiments to test the effectiveness. At last, we implement this triangular spectral-element method to solve the steady Stokes problem.

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1. Introduction

The spectral method, with its high accuracy, is becoming an attractive method nowadays. But the property of the basis function restricts its further application to problems on the general domain. The spectral-element method (SEM), which combines the high accuracy of the spectral method with the geometric flexibility of the finite element, greatly pushes the classical spectral method forward and has become a popular method for simulations of fluid dynamics, atmospheric modeling and many other challenging problems [2,3,5,6,13,15,20,23]. In order to extend the spectral-element method to problems on the general domain, Guo and Jia [7] study the spectral method on the convex quadrilateral; Jia and Guo [12] extend this method to the quadrilateral spectral-element method (QSEM) on the polygon; Guo and Jia [8] study the pseudospectral method on the quadrilateral. In some cases, the triangular element can approximate the complex boundary better and has more flexibility, which makes the spectral method on triangle become another research hotspot [9–11,13,16,19,24–26]. Generally speaking, according to the class of functions used in the approximations, the triangular spectral (element) method has three categories: (i) nodal basis methods based on high-order polynomial interpolation on special interpolation points, such as Fekete points; (ii) modal basis methods based on the Koornwinder–Dubiner(KD) polynomials; and (iii) approximations by non-polynomial functions on a triangle. Taylor and co-authors [17,28,29] study the high accuracy integral formula on some

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* Corresponding author.
E-mail addresses: hpma@shu.edu.cn, fxljl@just.edu.cn (H. Ma).

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special nodes on a triangle. Karniadakis and Sherwin [13] use the Duffy transformation to generate warped tensorial orthogonal polynomials on triangles, and Li and Shen [16] analyze its optimal error estimate. Shan and Li [22] furthermore extend the polynomial spectral method on a single triangle to the triangular spectral-element method (TSEM) for the eigenvalue problems of Stokes equation. In [11,24], the Duffy transformation is also used to generate rational basis functions rather than polynomials. Chen et al. [4] apply the method in [24] to the Navier–Stokes problem. It is known that the Duffy transformation is not one-to-one. It collapses one edge of the reference square into a vertex of the triangle, so the computational grids severely gather near the singular vertex. Moreover, the integral singularity brought by the Duffy transformation in computation of the stiffness matrix using the nodal basis seems hard to handle. To break through these drawbacks, the paper [19] puts forward a new one-to-one transformation between the square and the triangle, which pulls one edge (at the middle point) of the triangle to two edges of the square. Based on this transformation, Samson et al. [21] study a triangular spectral method using the modal basis, where a recursive method is proposed to handle the singularity. The theoretical analysis and numerical tests therein show the efficiency of the method, but it seems not convenient to extend the method to TSEM. In the presented paper, also based on the one-to-one transformation in [19], we study a new nodal basis TSEM. This TSEM enjoys the following advantages: firstly, the grid distribution is more uniform than that produced by the commonly-used Duffy transformation; secondly, almost fully enjoying the tensorial product property of the nodal basis makes this TSEM convenient to handle the nonlinear or variable coefficient problems; thirdly, the integral singularity brought by the one-to-one mapping can be removed in a naive and stable way; at last, the codes of this TSEM can be obtained through slight revision of the codes commonly used in standard nodal basis QSEM.

In Section 2, we present the one-to-one transformation used in the presented TSEM. In Section 3, we introduce the approximation space and study a quasi-interpolation operator. In Section 4, we put forward a new triangular spectral-element method for the modal elliptic problem; we present the approximation scheme, the analysis of convergence and do three tests. In Section 5, we implement this TSEM to the steady Stokes problem. We make a conclusion in the last section.

2. Preliminaries

In this section, we present the related properties of the one-to-one transformation $\boldsymbol{F}_\triangle$ used in this TSEM. After that we have a look at the integral singularity.

2.1. One-to-one transformation between square and triangle

Let (ξ, η) be the coordinate system on the reference square $\square := \Lambda_\xi \times \Lambda_\eta = (-1,1)^2 = \Lambda^2$, and (x, y) be the coordinate system related to the triangle $\triangle := \triangle Q_1Q_2Q_3$ with vertex coordinates $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Throughout this paper, we use boldface letters to denote vectors or vector-valued functions, e.g.,

$$\boldsymbol{x} = (x, y); \quad \boldsymbol{x}_j = (x_j, y_j), \quad 1 \le j \le 3; \quad \boldsymbol{a}_j = (a_j, b_j), \quad 1 \le j \le 4. \tag{2.1}$$

Then the one-to-one mapping from the reference square to the triangle, $\boldsymbol{F}_\triangle \colon \square \to \triangle$, is given by

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x}_1 \frac{(1+\xi)(3-\eta)}{8} + \boldsymbol{x}_2 \frac{(1-\xi)(1-\eta)}{4} + \boldsymbol{x}_3 \frac{(3-\xi)(1+\eta)}{8} \\ &= \boldsymbol{a}_1 + \boldsymbol{a}_2\xi + \boldsymbol{a}_3\eta + \boldsymbol{a}_4\xi\eta, \qquad (\xi,\eta) \in \bar{\square}.\end{aligned} \tag{2.2}$$

Easy to see, the transformation (2.2) maps the midpoint of the edge Q_1Q_3 to the vertex (1,1) of $\square$. By the direct calculation, we have

$$\begin{aligned}\boldsymbol{a}_1 &= (3\boldsymbol{x}_1 + 2\boldsymbol{x}_2 + 3\boldsymbol{x}_3)/8, \quad \boldsymbol{a}_2 = (3\boldsymbol{x}_1 - 2\boldsymbol{x}_2 - \boldsymbol{x}_3)/8, \\ \boldsymbol{a}_3 &= (-\boldsymbol{x}_1 - 2\boldsymbol{x}_2 + 3\boldsymbol{x}_3)/8, \quad \boldsymbol{a}_4 = (-\boldsymbol{x}_1 + 2\boldsymbol{x}_2 - \boldsymbol{x}_3)/8.\end{aligned} \tag{2.3}$$

The Jacobian matrix and the Jacobian determinant of the transformation (2.2) are

$$\mathbb{J}_\triangle = \begin{bmatrix} \partial_\xi \boldsymbol{x} \\ \partial_\eta \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_2 + \boldsymbol{a}_4\eta \\ \boldsymbol{a}_3 + \boldsymbol{a}_4\xi \end{bmatrix}, \quad J_\triangle = \det(\mathbb{J}_\triangle) = \frac{S}{8}(2 - \xi - \eta), \tag{2.4}$$

where $S = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \end{vmatrix} > 0$ is the area of the triangle $\triangle$.

We now provide an insight about the transformation $\boldsymbol{F}_\triangle$ and its Jacobian determinant $J_\triangle$ from another perspective. Recall that the usual transformation $\boldsymbol{F}_Q$ (cf. [1,3,13,27,30]) between $\square$ and the convex quadrilateral Q is

$$\boldsymbol{x} = \boldsymbol{x}_1 \frac{(1+\xi)(1-\eta)}{4} + \boldsymbol{x}_2 \frac{(1-\xi)(1-\eta)}{4} + \boldsymbol{x}_3 \frac{(1-\xi)(1+\eta)}{4} + \boldsymbol{x}_4 \frac{(1+\xi)(1+\eta)}{4}, \quad (\xi,\eta) \in \bar{\square}, \tag{2.5}$$

where $\boldsymbol{x}_i, i = 1,2,3,4$ are coordinates of the four vertices $Q_i, i = 1,2,3,4$ of the quadrilateral Q. In fact, the transformation (2.2) is a special form of the transformation (2.5) when Q_4 coincides with the midpoint of Q_1Q_3. To demonstrate this, assume that Q_4 locates on Q_1Q_3 and

$$\boldsymbol{x}_4 = (1-\lambda)\boldsymbol{x}_3 + \lambda\boldsymbol{x}_1, \quad \lambda \in [0,1]. \tag{2.6}$$

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