Applied Mathematics and Computation 321 (2018) 255–266

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An effective computational method for solving linear multi-point boundary value problems

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ARTICLE INFO

Keywords:
Multi-point boundary value problems
Least squares approximation method
Lagrange-multiplier method
Approximate series solutions
Residual error function

ABSTRACT

In this work, an efficient computational method is proposed for solving the linear multi-point boundary value problems (MBVPs). Our approach depends mainly on of the least squares approximation method, the Lagrange-multiplier method and the residual error function technique. With the proposed scheme, we handle the numerical solutions of the linear MBVPs in a straightforward manner. Firstly, the given linear MBVP is reduced to a linear system of algebraic equations, and the coefficients of the approximate polynomial solution of the problem are determined by solving this system. Secondly, a linear boundary value problem related to the error function of the approximate solution is constructed, and error estimation is presented for the suggested method. The convergence of the approximate solution is proved. The reliability and efficiency of the proposed approach are demonstrated by some numerical examples.

1. Introduction

Multi-point boundary value problems are closely related to various practical problems in different areas of science and engineering, such as the vibrations of a guy wire of uniform cross section [1], the theory of elastic stability [2], the large bridges design problem [3,4], and so on. In the past twenty years, the problem of existence and uniqueness of the solutions for the MBVPs has been extensively investigated; see [5–13] and the references therein. Meanwhile, quite a few methods have been proposed to handle the numerical solutions of the MBVPs, such as the shooting method [4,14], the finite difference method [15], the Adomian decomposition method [16], the variational iteration method [17], the homotopy analysis method [18], the Sinc-collocation method [19], the optimal homotopy asymptotic method [20], the shifted Jacobi spectral method [21], the reproducing kernel method and its modifications [22–31], and the Pade approximant [32]. More recently, an effective algorithm based mainly on differential transform method was proposed to solve the general n order MBVPs [33].

The squared reminder minimization method (SRMM) was introduced by Bota and Căruntu [34] for solving the multi-pantograph equation, and this method was developed in [35–39] for the numerical solutions of various kinds of nonlinear differential and integral equations. In [40], the authors proposed a similar approach named as the best square approximation method (BSAM) to solve a mixed linear Volterra–Fredholm integral equation. Recently, Wang et al. used the least squares approximation method (LSAM), which can be considered as a developed version of the SRMM and the BSAM, for solving

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https://doi.org/10.1016/j.amc.2017.10.016

the Volterra–Fredholm integral equations [41]. In [42], the authors applied the LASM to handle the numerical solutions of Hammerstein–Volterra delay integral equations.

The motivation of this study is to combine the LASM, the Lagrange-multiplier method and the residual error function technique, to obtain the approximate solution of the following linear differential equation

$$\mathbb{L}[u(x)] = u^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)u^{(i)}(x) = f(x), \quad 0 \le x \le 1, \tag{1}$$

subject to the boundary conditions

$$\sum_{j=1}^{n}\sum_{k=0}^{n-1} \alpha_{jk}^{m} u^{(k)}(\xi_j) = \beta_m, \quad m = 1, 2, \ldots, n, \tag{2}$$

where $0 \le \xi_1 < \xi_2 < \cdots < \xi_n \le 1$, α_{jk}^m and β_m are real constants. We assume that f has the properties which guarantee the existence and uniqueness of the solution of the problem (1) under the conditions (2). The basic ideas of the previous works [34–42] are developed and applied to the problems (1) and (2).

The residual error function technique in our proposed approach is used to handle error estimation of the numerical solutions. The residual error function was first introduced by Oliveira [43] and this technique was successfully used by several researchers to measure the absolute errors of numerical solutions of various kinds of function equations [44–50].

The rest of the paper is organized as follows. Section 2 is devoted to the explanation of the proposed method for solving the problems (1) and (2). Convergence analysis and an error estimation are presented in Section 3. Section 4 shows some numerical examples to testify the validity and applicability of the proposed method. In Section 5, we end this paper with a brief conclusion.

2. Method of solution

Firstly, we define the operator

$$(\mathbb{D}u)(x) = u^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)u^{(i)}(x) - f(x). \tag{3}$$

Suppose that $\phi_0(x), \phi_1(x), \ldots, \phi_N(x)$ are linearly independent functions on the close interval [0, 1], and $\Phi_N = \text{span}\{\phi_0(x), \phi_1(x), \ldots, \phi_N(x)\}$ is generated by the linear space. Therefore, for any $u_N(x) \in \Phi_N$, there exist $c_0, c_1, \ldots, c_N$ such that

$$u_N(x) = \sum_{l=0}^{N} c_l \phi_l(x). \tag{4}$$

Here, we want to find the approximate solution $u_N(x)$ defined in (4) of the problems (1) and (2). The solution $u_N(x)$ need to satisfy the following conditions:

$$|(\mathbb{D}u_N)(x)| < \epsilon \tag{5}$$

and

$$\sum_{j=1}^{n}\sum_{k=0}^{n-1} \alpha_{jk}^{m} u_N^{(k)}(\xi_j) = \beta_m, \quad m = 1, 2, \ldots, n. \tag{6}$$

In order to determine the coefficients $c_0, c_1, \ldots, c_N$ of the approximate solution $u_N(x)$, we substitute (4) into (3) such that

$$\begin{aligned}(\mathbb{D}u_N)(x) &= \sum_{l=0}^{N} c_l \phi_l^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)\left(\sum_{l=0}^{N} c_l \phi_l^{(i)}(x)\right) - f(x) \\ &= \sum_{l=0}^{N} c_l \left(\phi_l^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)\phi_l^{(i)}(x)\right) - f(x) \\ &= \sum_{l=0}^{N} c_l \alpha_l(x) - f(x),\end{aligned} \tag{7}$$

where $\alpha_l(x) = \phi_l^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x)\phi_l^{(i)}(x), \quad l = 0, 1, \ldots, N.$

In the following, we introduce a real function:

$$J = J(c_0, c_1, \ldots, c_N) = \int_0^1 (\mathbb{D}u_N)^2(x)\mathrm{d}x. \tag{8}$$