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# Chaotic congestion games

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ABSTRACT

We analyze a class of congestion games where two agents must send a finite amount of goods from an initial location to a terminal one. To do so the agents must use resources which are costly and costs are load dependent. In this context we assume that the agents have limited computational capability and they use a gradient rule as a decision mechanism. By introducing an appropriate dynamical system, which has the steady state exactly at the unique Nash equilibrium of the static congestion game, we investigate the dynamical behavior of the game. We provide a local stability condition in terms of the agents' reactivity and the nonlinearity of the cost functions. In particular we show numerically that there is a route to complex dynamics: a cascade of flip-bifurcation leading to periodic cycles and finally to chaos.



## 1. Introduction

Congestion games are a very important class of games which have been extensively investigated. The applicability of congestion games and their uses are such that a large amount of research is still ongoing and new directions are being explored and investigated. We can describe a congestion game as a game where there is a ground of resources and each player can allocate a subset of them. Of course no resource is free so costs are to be paid. The central point is that each resource has a cost which depends on the load induced by the players who use it. Of course, each player tries to do his/her best but ends up paying the cost of congestion due to other people's decisions. These type of games are quite simple but have enough structure to model very interesting strategic situations spanning from oligopoly, migrations and the internet (see [17]).

Moreover, as established in the landmark paper of Rosenthal (see [16]), they have always a Nash equilibrium in pure strategies. While Nash equilibria have traditionally been studied as they have some very desirable properties it is also true that recently an effort on understanding the computational cost of such a type of equilibria has produced very surprising results. For this reason, in this context, equilibria which are computationally less demanding, like for example correlated equilibria, have been investigated [18]. This is where our investigation really starts and our contribution lies.

We explicitly move away from the highly rational framework of Nash equilibrium and its variations and we posit for our agent different decision rules. Namely, by moving from a static framework to a dynamic one, we assume that our agents are adopting a gradient rule to react to other people's decisions and we introduce an important parameter in order to control the

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level of reactivity of our agents. Such a mechanism has been extensively studied in the literature in the context of oligopolist competition and rent-seeking games as the reader can see [3–7,21]. Moreover, such a mechanism has been investigated in relationship to the role of heterogeneity in the named games as the reader can see [1,2,8,9,20] and also in other contexts as the prisoner's dilemma and public good games by Perc and coauthors (see [12–15]). Of course in our dynamical system the Nash equilibrium is the unique steady state. We study the dynamics generated by this adjustment process explicitly. Our first finding is that the reactivity has a destabilizing role as always happens in this strand of literature. We also show that the volume of resources which must travel on the network and the nonlinearity of the cost functions as well as their asymmetry greatly influence the behavior as destabilizing forces and therefore we find, in a different setting, that the role of nonlinearity is still prominent for this class of games. Previously the role of nonlinearity was investigated in relation to the loss of optimality or cost of anarchy in the static context which is, in our setting, our steady state. We stress that the destabilizing role of those forces act via a period-doubling bifurcation mechanism. Interestingly, we find that our dynamics could shed some light, at a qualitative level, about results obtained in a laboratory (see [19]). Such experimental evidence points explicitly to the persistence of irregular fluctuations around the Nash equilibrium. Our model exhibits consistently and robustly the presence of fluctuations, near the Nash equilibrium, which are not normally distributed as observed also in the experimental and empirical literature.

The paper is organized as follows. In the second section we present the relevant definitions of congestion games and we will characterize the equilibrium and then some of its properties related to the nonlinearity of the costs functions and the amount of goods which must travel on the network. Hence, in the third section, we study the dynamic setting and we explicitly find analytic conditions to characterize stability and its relation to the agents' reactivity parameters. In the fourth section several numerical simulations are presented in order to illustrate our results.

## 2. Congestion games

To begin we briefly present the most important definitions. We work with a very simple network. There are two nodes, the source and the target, that we call $S$ and $T$ and there are two possible paths that connect them, we call the paths $e$ and $f$ and we refer to them as resources. The set of resources is $\mathcal{E}=\{e, f\}$ which must be shared among two players and each resource $e\in\mathcal{E}$ has a load-dependent cost given by $c_e:\mathbb{R}_+\to\mathbb{R}_+$, we denote with $\mathcal{L}$ the set of all cost functions. Each player $i=1,2$ must send $d_i$ units from the node $S$ to the node $T$, we can summarize the choice by giving the pairs $\left(x_e^i, x_f^i\right)$ with

$$x_e^i + x_f^i = d_i$$

The set of cost functions is denoted by $\mathcal{L}$ and we assume that if $\ell\in\mathcal{L}$ then the function $x\ell(x)$ is convex, in this case we say that $\ell$ is semi-convex. We assume that each element of $\mathcal{L}$ is differentiable, non-decreasing and semi-convex. Given a strategy profile $x=\left(x^i\right)_{i\in N}$ and a resource $e\in\mathcal{E}$ we call

$$\sum_{i\in N} x_e^i \overset{def}{=} x_e \qquad \text{the total load on } e$$

Given a strategy profile $x=\left(x^i\right)_{i\in N}$ the cost of player $i\in N$ is defined as

$$c_i(x) = \sum_{e\in\mathcal{E}} x_e^i \cdot \ell_e(x_e)$$

and each player tries to maximize $-c_i(x)$. The social cost is, by definition,

$$c(x) = \sum_{i\in N} c_i(x) = \sum_{i\in N}\sum_{e\in\mathcal{E}} x_e^i \cdot \ell_e(x_e)$$

Of course the most important thing are the equilibria of the game. A pure strategy Nash equilibrium is a strategy profile $x=\left(x^i\right)_{i\in N}$ such that for each agent the following

$$c_i(x) \le c_i(y^i, x^{-i}) \qquad \text{for every } y^i\in S_i$$

It is important to keep in mind that for this class of games it is possible to characterize a Nash equilibrium in a very elegant way. In fact it has been proven that the game has the Nash equilibrium $x=\left(x^i\right)_{i\in N}$ if and only if the variational inequality

$$\sum_{e\in\mathcal{E}} \ell_e(x_e)(y_e^i - x_e^i) \ge 0$$

is satisfied for any player and for any strategy (see [10]).

It is well known that a Nash equilibrium requires a very high level of rationality and coordination and therefore a considerable amount of effort has been spent on relaxing this stringent assumption. For this reason a large body of literature has been developed in order to work with correlated equilibria and similar concepts. In this paper, as we already explained