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New perturbation bounds of partitioned generalized Hermitian eigenvalue problem[☆]

Chuanfu Xiao, Hanyu Li[*]

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, PR China

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ABSTRACT

In this paper, some new absolute perturbation bounds of partitioned generalized Hermitian positive definite eigenvalue problem are established by two different ways. Numerical results show that our bounds are sharper than the ones in the literature.

1. Introduction

There are many classical results on perturbation analysis for generalized Hermitian eigenvalue problem; see e.g., [1–3]. However, most of these results are built on chordal metric [1–3]. Considering that the chordal metric is not invariant under scaling and the related bounds are less intuitive than the ones built on Euclidean metric, Nakatsukasa investigated the perturbation bounds for generalized Hermitian eigenvalue problem based on Euclidean metric; see e.g., [4–6]. In this paper, we continue the research on perturbation bounds for generalized Hermitian eigenvalue problem along the lines of [4–6]. More specifically, we consider the absolute perturbation bounds of the partitioned Hermitian positive definite pair (A, B), that is,

$$A = \begin{pmatrix} A_{11} & A_{21}^H \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{21}^H \\ B_{21} & B_{22} \end{pmatrix}, \tag{1.1}$$

where $A_{11},\ B_{11} \in \mathbb{C}^{k\times k},\ A,\ B \in \mathbb{C}^{n\times n}$ are Hermitian and B is Hermitian positive definite. Now let $\hat{A} = A + E$ and $\hat{B} = B + F$, where

$$E = \begin{pmatrix} E_{11} & E_{21}^H \\ E_{21} & E_{22} \end{pmatrix} \in \mathbb{C}^{n\times n}, \quad F = \begin{pmatrix} F_{11} & F_{21}^H \\ F_{21} & F_{22} \end{pmatrix} \in \mathbb{C}^{n\times n} \tag{1.2}$$

are Hermitian and F is such that $B + F$ is Hermitian positive definite, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n$ be the generalized eigenvalues of (A, B) and $(\hat{A}, \hat{B})$, respectively. Our goal is to evaluate the upper bound of $|\hat{\lambda}_i - \lambda_i|$ for any $i = 1, 2, \ldots, n$. Note that there are no infinite eigenvalues for the above partitioned Hermitian positive definite pair, because $B + F$ is assumed to be positive definite. So we can set the generalized eigenvalues to be in the above form.

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* Corresponding author.
E-mail addresses: hyli@cqu.edu.cn, lihy.hy@gmail.com (H. Li).

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For the above generalized Hermitian eigenvalue problem, Li et al. [7] considered the off-diagonal perturbation and Xu and Liu [8] extended the results in [7] to general case. However, their results contain the unknown variable $\hat{\lambda}_i$. Some perturbation bounds without $\hat{\lambda}_i$ can be found in [4,5], but they are a little loose. In this paper, we will first provide a new bound without $\hat{\lambda}_i$ using a method based on calculus. Numerical experiments show that this bound is sharper than the ones in [4,5]. Furthermore, we also obtain two bounds for the above generalized Hermitian eigenvalue problem by transforming the problem to the standard Hermitian eigenvalue problem and using the result from [9], and do some numerical experiments on these two bounds.

2. Main result 1

In this section, we will establish a perturbation bound without the unknown variable $\hat{\lambda}_i$ for the partitioned generalized Hermitian eigenvalue problem mentioned in Section 1. Some preliminaries are first introduced as follows.

Lemma 2.1 (See [10]). *Let $A, B, E, F \in \mathbb{C}^{n\times n}$ be Hermitian matrices and suppose that $(A+tE, B+tF)$ is Hermitian positive definite pair for any $t\in[0, 1]$. Define the eigenvector function $x(t)$ by $(A+tE)x(t) = \lambda_i(t)(B+tF)x(t)$, where $\|x(t)\|_2 = 1$. If $\lambda_i(t)$ is the ith simple generalized eigenvalue of $(A+tE, B+tF)$, then*

$$\frac{\partial\lambda_i(t)}{\partial t} = \frac{x^H(t)Ex(t) - \lambda_i(t)x^H(t)Fx(t)}{x^H(t)(B+tF)x(t)}. \tag{2.1}$$

Lemma 2.2 (See [5,6]). *Suppose that A, B, E and F are the same as Lemma 2.1 and let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\hat{\lambda}_1 \ge \hat{\lambda}_2 \ge \cdots \ge \hat{\lambda}_n$ be the generalized eigenvalues of (A, B) and $(\hat{A}, \hat{B})$ with $\hat{A} = A+E$ and $\hat{B} = B+F$, respectively. Then*

$$|\hat{\lambda}_i - \lambda_i| \le \|(B+F)^{-1}\|_2\|E - \lambda_i F\|_2, \tag{2.2}$$

and

$$|\hat{\lambda}_i - \lambda_i| \le \|B^{-1}\|_2\|E - \hat{\lambda}_i F\|_2, \tag{2.3}$$

for all $1\le i\le n$.

Lemma 2.3. *Let A, B, E, and F be defined as in (1.1) and (1.2) and let $\lambda_1(t)\ge\lambda_2(t)\ge\cdots\ge\lambda_n(t)$ be the generalized eigenvalues of $(A+tE, B+tF)$ for any $t\in[0, 1]$. Suppose that $x(t)$ is the unit generalized eigenvector corresponding to $\lambda_i(t)$ and is partitioned as $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}$ according to the block matrices A, B, E, and F. Denote $c_1 = \lambda_{\min}(B_{11})$, $c_2 = \lambda_{\min}(B_{22})$, $c = \lambda_{\min}(B)$ and the gaps $\phi_i = \min_{\lambda\in\lambda(A_{11},B_{11})}|\lambda_i - \lambda|$, $\varphi_i = \min_{\lambda\in\lambda(A_{22},B_{22})}|\lambda_i-\lambda|$, where $\lambda_i\in\lambda(A, B)$. Let $\lambda_i(t)$ be the ith simple generalized eigenvalue of $(A+tE, B+tF)$ for all $t\in[0, 1]$. If $c > \|F\|_2$, and*

$$c_1\phi_i > \frac{\|E-\lambda_iF\|_2}{c-\|F\|_2}\|B_{11}\|_2 + \|E_{11}\|_2 + \frac{|\lambda_i|c+\|E\|_2}{c-\|F\|_2}\|F_{11}\|_2 \tag{2.4}$$

and

$$c_2\varphi_i > \frac{\|E-\lambda_iF\|_2}{c-\|F\|_2}\|B_{22}\|_2 + \|E_{22}\|_2 + \frac{|\lambda_i|c+\|E\|_2}{c-\|F\|_2}\|F_{22}\|_2 \tag{2.5}$$

are satisfied simultaneously, then

$$\|x_1(t)\|_2 \le \frac{\alpha(\|B_{21}\|_2+\|F_{21}\|_2) + (\|A_{21}\|_2+\|E_{21}\|_2)}{\sqrt{\left(c_1\phi_i - \left(\frac{\|E-\lambda_iF\|_2}{c-\|F\|_2}\|B_{11}\|_2 + \|E_{11}\|_2 + \alpha\|F_{11}\|_2\right)\right)^2 + (\alpha(\|B_{21}\|_2+\|F_{21}\|_2) + \|A_{21}\|_2+\|E_{21}\|_2)^2}} \tag{2.6}$$

and

$$\|x_2(t)\|_2 \le \frac{\alpha(\|B_{21}\|_2+\|F_{21}\|_2) + (\|A_{21}\|_2+\|E_{21}\|_2)}{\sqrt{\left(c_2\varphi_i - \left(\frac{\|E-\lambda_iF\|_2}{c-\|F\|_2}\|B_{22}\|_2 + \|E_{22}\|_2 + \alpha\|F_{22}\|_2\right)\right)^2 + (\alpha(\|B_{21}\|_2+\|F_{21}\|_2) + \|A_{21}\|_2+\|E_{21}\|_2)^2}} \tag{2.7}$$

are true for all $t\in[0, 1]$, where $\alpha = \frac{|\lambda_i|c+\|E\|_2}{c-\|F\|_2}$, and $\lambda_i(0) = \lambda_i$, $\lambda_i(1) = \hat{\lambda}_i$ for any $i = 1, 2, \ldots, n$.

Proof. It is easy to check that

$$\|(A_{11}-\lambda_iB_{11})x_1(t)\|_2 - (|\lambda_i(t)-\lambda_i|\|B_{11}\|_2 + t\|E_{11}-\lambda_i(t)F_{11}\|_2)\|x_1(t)\|_2 \le \|((A_{11}+tE_{11})-\lambda_i(t)(B_{11}+tF_{11}))x_1(t)\|_2, \tag{2.8}$$

and

$$\|(A_{11}-\lambda_iB_{11})x_1(t)\|_2 = \|(P_1^H)^{-1}P_1^H(A_{11}-\lambda_iB_{11})P_1P_1^{-1}x_1(t)\|_2 \ge c_1\phi_i\|x_1(t)\|_2, \tag{2.9}$$