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Strong edge chromatic index of the generalized Petersen graphs

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ABSTRACT

A strong edge coloring of a graph G is an assignment of colors to the edges of G such that two distinct edges are colored differently if they are adjacent to a common edge or share an endpoint. The strong chromatic index of a graph G, denoted by $\chi'_s(G)$, is the minimum number of colors needed for a strong edge coloring of G. We determine the strong chromatic index of the generalized Petersen graphs $P(n, k)$ when $1 \le k \le 3$.

1. Introduction

All graphs considered in this paper are finite and simple. Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E, $|V|$ and $|E|$ are its *order* and *size*, respectively. The girth of G is the length of the shortest cycle in G. The maximum degree of G is denoted by $\Delta(G)$ and simply by Δ. A *strong k-edge coloring* of a graph G is a mapping $\phi : E \mapsto \{1, \ldots, k\}$ such that any two edges are colored differently if they are adjacent to a common edge or share an endpoint. In other words, the edges in each color class form an induced matching in G. The *strong chromatic index* of a graph G, denoted $\chi'_s(G)$, is the minimum number of colors needed for a strong edge coloring of G.

The notion of strong edge chromatic index was first introduced by Fouquet and Jolivet [10,11]. The strong edge coloring problem is NP-complete even for bipartite graphs at least 4 [19]. However, polynomial time algorithms have been obtained for chordal graphs [5], co-comparability graphs [13], and partial k-trees [21].

In 1998, Erdős [8] asked whether it is true that $\chi'_s(G) \le \frac{5}{4}\Delta^2$. Faudree et al. [9] revised the above question as follows:

Conjecture 1. *(Faudree et al. [10]).*

$$\chi'_s(G) \le \begin{cases} \frac{5}{4}\Delta^2 & if\ \Delta\ is\ even, \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & if\ \Delta\ is\ odd, \end{cases}$$

It is clear that Conjecture 1 holds when $\Delta \le 2$. In regard to subcubic graphs, i.e., graphs with maximum degree at most 3, Faudree et al. [10] posed the following set of conjectures.

Conjecture 2. *(Faudree et al. [10]). Let G be a subcubic graph.*

2.1 $\chi'_s(G) \le 10$.

2.2 *If G is bipartite, then* $\chi'_s(G) \le 9$.

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2.3 *If G is planar, then* $\chi'_s(G) \le 9$.
2.4 *If G is bipartite and the degree sum along every edge is at most 5, then* $\chi'_s(G) \le 6$.
2.5 *If G is bipartite with girth at least 6, then* $\chi'_s(G) \le 7$.
2.6 *If G is bipartite with large girth, then* $\chi'_s(G) \le 5$.

Conjecture 2.1 was proved to be true for $\Delta = 3$ by Andersen [2] and, independently, by Hork et al. [14]. Conjecture 2.2 was verified by Steger and Yu [23]. Conjecture 2.4 was confirmed by Wu and Lin [26] and was generalized by Nakprasit and Nakprasit [20]. Borodin and Ivanova [4] verified Conjecture 2.6 for planar graphs.

Shiu and Tam [22] determined the strong chromatic index of the complete cubic Halin graph. Lih and Liu [17] proved that a cubic Halin graph G is different from two particular graphs then $\chi'_s(G) \le 7$. Lai et al. [15] proved that a Halin graph $G = T \cup C$ is different from a certain necklace Ne_2 and any wheel W_n ($n \not\equiv 0$ mod 3) then $\chi'_s(G) \le \chi'_s(T) + 3$.

Throughout the paper, all subscripts are taken modulo n. Given integers $n \ge 3$ and $1 \le k < \frac{n}{2}$, the *generalized Petersen graph* $P(n, k)$ has $2n$ vertices labeled $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$,. The vertices labeled $v_1, v_2, \ldots, v_n$ are called *outer vertices* while those labeled $u_1, u_2, \ldots, u_n$ are *inner vertices*. There are three types of edges in $P(n, k)$: (a) *outer edges*, connecting outer vertices in the form v_iv_{i+1} for each $i \in \{1, 2, 3, \ldots, n\}$; ($b$) *inner edges*, connecting inner vertices in the form u_iu_{i+k}, for each $i \in \{1, 2, 3, \ldots, n\}$; and ($c$) *spokes*, or edges connecting an outer vertex with an inner one, in the form v_iu_i for $i = 1, 2, 3, \ldots, n$. These graphs were introduced by Coxeter [6] and named by Watkins [24] based on the fact that $P(5, 2)$ is the Petersen graph. For two positive integers n and k, (n, k) denotes the greatest common divisor. Various properties of the generalized Petersen graphs are extensively studied, see for some recent results [1,3,7,12,16,18,25,27,28].

Motivated from Conjectures 2.5 and 2.6, in this paper, we consider the strong chromatic number of the generalized Petersen graphs. First we consider when a generalized Petersen graph is bipartite.

Theorem 1.1. *For two integers n and k with* $n > 2k$ *and* $k \ge 1$, $P(n, k)$ *is bipartite if and only if* $\frac{n}{(n,k)}$ *is even and k is odd.*

Proof. It is well known that a graph is bipartite if and only if it contains no odd cycle. We show its necessity based on this criterion. Assume that $P(n, k)$ is bipartite. Observe that there is a cycle of length $k+3$ in $P(n, k)$, for instance $v_1, v_2, \ldots, v_{k+1}, u_{k+1}, u_1, v_1$, implying that k is odd. Moreover, there is a cycle of length $\frac{n}{(n,k)}$ in $P(n, k)$, for instance $u_1, u_{k+1}, \ldots, u_{lk+1}, u_1$, where $l = \frac{n}{(n,k)} - 1$ implying that $\frac{n}{(n,k)}$ is even.

Next we show its sufficiency by giving a proper 2-vertex-coloring of $P(n, k)$. Since $\frac{n}{(n,k)}$ is even, n is even. Let c: $V(P(n, k)) \mapsto \{1, 2\}$ be defined as follows: $c(v_i) = 1$ if i is odd, $c(v_i) = 2$ otherwise; $c(u_i) = 1$ if i is even, $c(u_i) = 2$ otherwise. We claim that c is proper. By the definition of c, it remains to show that for any edge $e = u_iu_j$, $c(u_i) \ne c(u_j)$. Without loss of generality, let $j = i + k$. Since k is odd, i and j have different parity. By the definition of c, we have $c(u_i) \ne c(u_j)$. So, c is proper, implying that G is bipartite. □

In this paper, we determine the strong chromatic index of generalized Petersen graphs $P(n, k)$ when $1 \le k \le 3$. For convenience, a k-edge coloring c of $G = (V, E)$ is given in the form of a partition $\{E_1, E_2, \ldots, E_k\}$ of E, where E_i denotes the (possibly empty) of set of edges of assigned color i.

2. $\chi'_s(P(n, 1))$

Note that $P(n, 1) \cong C_n \Box K_2$, the cartesian product of the cycle C_n of order n and K_2.

Theorem 2.1. *For an integer* $n \ge 3$,

$$\chi'_s(P(n,1)) = \begin{cases} 9, & \text{if } n = 3, \\ 8, & \text{if } 5 \le n \le 6, \\ 6, & \text{if } n \text{ is divisible by 4}, \\ 7, & \text{otherwise}. \end{cases}$$

Proof. It is straightforward to check that $\chi'_s(P(3, 1)) = 9$ and $\chi'_s(P(n, 1)) = 8$ for $n \in \{5, 6\}$ (see a strong edge coloring of $P(n, 1)$ as shown in Fig. 2.1 using $\chi'_s(P(n, 1))$ colors). □

Assume that n is divisible by 4. Since v_1v_2, v_2v_3, v_3v_4, v_2u_2, v_3u_3, u_2u_3 must be assigned distinct colors, $\chi'_s(P(n, 1)) \ge 6$. On the other hand, we can find a strong 6-edge coloring (see Fig. 2.2) of $P(n, 1)$ with

$$E_1 = \left\{v_{1+4i}v_{2+4i}, u_{3+4i}u_{4+4i} : 0 \le i \le \frac{n-4}{4}\right\},$$

$$E_2 = \left\{v_{2+4i}v_{3+4i}, u_{4+4i}u_{5+4i} : 0 \le i \le \frac{n-4}{4}\right\},$$

$$E_3 = \left\{v_{3+4i}v_{4+4i}, u_{1+4i}u_{2+4i} : 0 \le i \le \frac{n-4}{4}\right\},$$

$$E_4 = \left\{v_{2i}u_{2i} : 1 \le i \le \frac{n}{2}\right\},$$