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Global well-posedness of incompressible Bénard problem with zero dissipation or zero thermal diffusivity

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ABSTRACT

In this paper, we establish the global well-posedness of the classical solution for the two-dimensional Bénard problem with zero dissipation or zero thermal diffusivity. Our work is partially motivated by the results about Boussinesq equations with zero dissipation or zero thermal diffusivity in Chae (2006) and Hou and Li (2005).

1. Introduction and main results

In this paper, we investigate the following Bénard problem in two-dimensions,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \mu\triangle u + \theta e_2, \\ \partial_t \theta + (u\cdot\nabla)\theta = \kappa\triangle\theta + u\cdot e_2, \\ \text{div } u = 0, \end{cases} \tag{1.1}$$

where the unknown variables $u = (u_1(x,y,t), u_2(x,y,t))$, $\theta(x,y,t)$ and $p(x, y, t)$ are velocity field, the scalar temperature variation and the scalar pressure in a gravity field in $\mathbb{R}^2\times\mathbb{R}^+$, respectively. $e_2 = (0,1)$ denotes the unit vector in vertical direction, θe_2 denote the buoyancy force and $u\cdot e_2$ denotes the vertical disturbed velocity component. The constant $\mu \geq 0$ is viscosity coefficient and $\kappa \geq 0$ is diffusivity coefficient. To solve Eq. (1.1), we impose the initial velocity field and temperature as follows,

$$u(x,y,0) = u_0(x,y),\ \ \theta(x,y,0) = \theta_0(x,y). \tag{1.2}$$

In the atmosphere and ocean, the Bénard problem plays an important role to deal with convective motions in a heat fluid, especially in the case of planar stationary flows. It describes that the temperature is a constant and the viscous flows lie between two horizontal walls, supposed that the quantity of heat is different between two horizontal walls and the change of heat is triggered only by conduction, but the convective motions will occur if the temperature gradient passes a certain critical value. The earliest experiments to demonstrate in a definitive manner the onset of thermal instability in fluids are those of Bénard in 1900. The Bénard problem describes the physical phenomenon adequately and a general treatment of the problem of thermal instability. (Please see [2,6,16,21]).

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If we ignore the thermal effects and remove the vertical disturbed velocity component $u \cdot e_2$, the Bénard system becomes the following Boussinesq equations,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \mu \triangle u + \theta e_2, \\ \partial_t \theta + (u \cdot \nabla)\theta = \kappa \triangle \theta, \\ \operatorname{div} u = 0. \end{cases} \tag{1.3}$$

For two-dimensional Boussinesq equations with full dissipation and thermal diffusivity, namely, $\mu > 0$ and $\kappa > 0$, the global well-posedness of solution has been obtained by Cannon and DiBenedetto in [4]. On the other hand, whether or not classical solutions of 2D inviscid Boussinesq equations ($\mu = 0$, $\kappa = 0$) can develop finite-time singularities is an important open problem. As an intermediate case, many mathematicians investigated the 2D Boussinesq equations with partial dissipation and heat conductivities. As the pioneer results in this field, Chae in [5], Hou and Li in [14] showed independently the global well-posedness for the 2D Boussinesq equations with zero-diffusion ($\mu > 0$, $\kappa = 0$) or zero-dissipation ($\mu = 0$, $\kappa > 0$). These results solved completely the open problem 3 imposed by Moffatt in [20]. These works were elaborated in [1,7,12,18,19] provided that the initial data lie in more general spaces. Motivated by the related works on Boussinesq equations in two-dimensions, the main purpose in this paper is to establish the global well-posedness of solution to the Bénard problems (1.1) and (1.2) with zero-dissipation or zero-diffusivity.

The main results in this paper are stated as follows.

Theorem 1.1. *Suppose that $u_0, \theta_0 \in H^m(\mathbb{R}^2)$ $(m \geq 3)$ and $\operatorname{div} u_0 = 0$, the Cauchy problems (1.1) and (1.2) with $\mu > 0$ and $\kappa = 0$ has a unique global classical solution (u, θ) with*

$$u, \theta \in L^\infty\left([0, T); H^m\left(\mathbb{R}^2\right)\right),$$

for any $T > 0$.

Theorem 1.2. *Suppose that $u_0 \in H^1(\mathbb{R}^2)$, the initial vorticity $\omega_0 \in L^4(\mathbb{R}^2)$, $\theta_0 \in H^2(\mathbb{R}^2)$ and $\operatorname{div} u_0 = 0$, the Cauchy problems (1.1) and (1.2) with $\mu = 0$ and $\kappa > 0$ has a unique global classical solution (u, θ) with*

$$u \in L^\infty\left([0, T); H^1\left(\mathbb{R}^2\right)\right), \quad \omega \in L^\infty\left([0, T); L^4\left(\mathbb{R}^2\right)\right), \quad \theta \in L^\infty\left([0, T); H^2\left(\mathbb{R}^2\right)\right),$$

for any $T > 0$, where $\omega = \nabla \times u = \partial_x u_2 - \partial_y u_1$.

Remark 1.3. For the Bénard problem with zero viscosity $(\mu = 0, \kappa > 0)$, the global well-posedness issues with lower-regularity have been considered by Danchin and Paicu in Theorem 3 in [8]. More precisely, suppose that the initial data

$$u_0 \in L^2\left(\mathbb{R}^2\right), \quad \omega_0 \in L^r\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right) \ (r > 2) \ \text{ and } \ \theta_0 \in L^2\left(\mathbb{R}^2\right) \cap B^{-1}_{\infty,1}\left(\mathbb{R}^2\right),$$

they showed that the unique global solution (u, θ) satisfies

$$\theta \in \mathcal{C}\left(\mathbb{R}^+; L^2\left(\mathbb{R}^2\right) \cap B^{1}_{\infty,1}\left(\mathbb{R}^2\right)\right) \cap L^2_{loc}\left(\mathbb{R}^+; H^1\left(\mathbb{R}^2\right)\right) \cap L^1_{loc}\left(\mathbb{R}^+; B^1_{\infty,1}\left(\mathbb{R}^2\right)\right),$$

$$u \in \mathcal{C}^{0,1}_{loc}\left(\mathbb{R}^+; L^2\left(\mathbb{R}^2\right)\right) \quad \text{and} \quad \omega \in L^\infty_{loc}\left(\mathbb{R}^+; L^r\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)\right).$$

Here, $B^s_{q,r}$ with $s \in \mathbb{R}$ and $q, r \in [1, \infty]$ denotes an inhomogeneous Besov space consists of functions f in the dual of usual Schwarz class satisfying

$$\|f\|_{B^s_{q,r}} = \|2^{js}\|\Delta_j f\|_{L^q}\|_{l^r} < \infty.$$

Based on their works, Wu and Xue extended the results to the global well-posedness for the 2D inviscid Bénard system with fractional diffusivity and Yudovich's type data in [24]. Furthermore we would like to refer some works [9,11,13,22,23] for Boussinesq equations with fractional diffusivity and [10,15,18,19,25] for incompressible fluids with partial viscosities.

Finally, the remainder of this paper is organized as follows. We will establish the uniform estimates and obtain the global well-posedness of the Bénard system with $\mu > 0$ and $\kappa = 0$ in Section 2. The global regularity of strong solution to the Bénard system with zero dissipation is obtained in Section 3.

2. Global well-posedness of Bénard problem without thermal diffusivity

In this section, we will establish the global regularity for the the Cauchy problems (1.1) and (1.2) with $\mu > 0$ and $\kappa = 0$.

First, we obtain the standard energy estimates of the Cauchy problems (1.1) and (1.2) with $\mu > 0$ and $\kappa = 0$. Taking the L^2 inner product of Eq. (1.1) with u and θ, after integrating by parts in $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{d}{d\tau}\|u(\tau)\|_2^2 + \mu\|\nabla u\|_2^2 = \int_{\mathbb{R}^2} u_2\theta dxdy \leq \frac{1}{2}\left(\|u(\tau)\|_2^2 + \|\theta(\tau)\|_2^2\right), \tag{2.1}$$

and

$$\frac{1}{2}\frac{d}{d\tau}\|\theta(\tau)\|_2^2 = \int_{\mathbb{R}^2} u_2\theta dxdy \leq \frac{1}{2}\left(\|u(\tau)\|_2^2 + \|\theta(\tau)\|_2^2\right), \tag{2.2}$$