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Traceability on 2-connected line graphs

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ABSTRACT

In this paper, we mainly prove the following: Let G be a connected almost bridgeless simple graph of order n sufficiently large such that $\overline{\sigma}_2(G)=\min\{d(u)+d(v) : uv \in E(G)\} \geq 2(\lfloor n/11\rfloor - 1)$. Then either $L(G)$ is traceable or Catlin's reduction of the core of G is one of eight graphs of order 10 or 11, where the core of G is obtained from G by deleting the vertices of degree 1 of G and replacing each path of length 2 whose internal vertex has degree 2 in G by an edge. We also give a new proof for the similar theorem in Niu et al. (2012) which has flaws in their proof.

1. Introduction

For graph-theoretic notation not explained in this paper, we refer the reader to [2]. We consider only finite and loopless graphs in this paper. A graph is called *multigraph* if it contains multiple edges. A graph without multiple edges is called a *simple graph* or simply a graph. A graph G is *traceable* if it has a Hamilton path, i.e., a spanning path. For a vertex x of G, $N_G(x)$ is the neighborhood of x in G, and $d_G(x)$ is the degree of x in G. For a vertex set $S \subseteq V(G)$, $N_G(S) = \cup_{x\in S} N_G(x)$. By $\delta(G)$ and $\Delta(G)$ we denote the minimum degree and the maximum degree of G, respectively. A graph is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. Similarly, a graph is *triangle-free* if it has no K_3. Define $D_i(G) = \{v \in V(G)|\ d_G(v) = i\}$, $D(G) = D_1(G) \cup D_2(G)$ and $D_{\geq 3}(G) = \cup_{i\geq 3} D_i(G)$. An edge $e = uv \in E(G)$ is called a *pendant edge* of G if $\min\{d(u), d(v)\} = 1$. The *girth* of G, denoted by g, is the length of a shortest cycle of G. The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. For a graph G, we define $\overline{\sigma}_2(G) = \min\{d(u) + d(v) : uv \in E(G)\}$ and $\sigma_2(G) = \min\{d(u) + d(v) : uv \notin E(G)\}$.

In [1,7], Harris et al. concerned how small the order of a claw-free, 2-connected nontraceable graph is. They presented two smallest claw-free, 2-connected nontraceable graphs, both of which have order 18 and size 24, and proved the following result.

Theorem 1. (*Harris and Mossinghoff [7] and Bullock* et al. *[1]) Let G be a 2-connected, claw-free graph with $|V(G)| < 18$, then G is traceable.*

A graph G is *almost bridgeless* if every cut edge of G is incident with a vertex of degree 1. For almost bridgeless graphs, in [13], Xiong and Zong proved the following result.

Theorem 2. (*Xiong and Zong [13]) Let G be a connected almost bridgeless simple graph of order n such that*

$$\overline{\sigma}_2(G) > 2(\lfloor n/10\rfloor - 1).$$

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0096-3003/

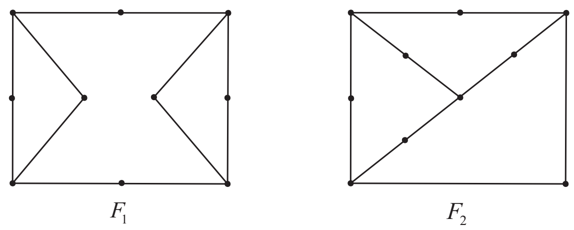

Fig. 1. Two graphs of order 10 that have no spanning trail.

If n is sufficiently large, then L(G) is traceable.

Let G be a connected multigraph. For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. Even when G is simple, G/X may not be simple. If Γ is a connected subgraph of G, then we write G/Γ for $G/E(\Gamma)$ and use v_Γ for the vertex in G/Γ to which Γ is contracted, and v_Γ is called a *contracted vertex* if $\Gamma \neq K_1$.

Let $O(G)$ be the set of vertices of odd degree in G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph Γ_R of G with $O(\Gamma_R) = R$. K_1 is regarded as a collapsible graph.

In [3], Catlin showed that every multigraph G has a unique collection of maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_c$. The *reduction* of G is $G' = G/(\cup_{i=1}^{c}\Gamma_i)$, the graph obtained from G by contracting each Γ_i into a single vertex v_i $(1 \le i \le c)$. For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $\Gamma_0(v)$ such that v is the contraction image of $\Gamma_0(v)$ and $\Gamma_0(v)$ is the *preimage* of v and v is a contracted vertex if $\Gamma_0(v) \neq K_1$. A graph G is *reduced* if $G' = G$.

For a graph G, in [9], Niu et al. considered the traceability of $L(G)$ about $\overline{\sigma}_2(G)$ and $\sigma_2(G)$, respectively. They proved the following results, where F_1, F_2 are depicted in Fig. 1.

Theorem 3. (*Niu et al. [9]*) *Let G be a connected almost bridgeless simple graph of order n such that*

$$\overline{\sigma}_2(G) \ge 2(\lfloor n/10 \rfloor - 1). \tag{1.1}$$

If n is sufficiently large, then either L(G) is traceable or the reduction of G equals F_1 or F_2.

Theorem 4. (*Niu et al. [9]*) *Let G be a 2-edge-connected simple graph with girth $g = 3$ or 4. If $\sigma_2(G) \ge \frac{2}{g-2}(\frac{n}{10} + g - 4)$ and n is sufficiently large, then either L(G) is traceable or the reduction of G equals F_1 or F_2.*

An edge cut X of G is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge-cut X with $|X| < k$. Note that a graph G is essentially k-edge-connected if and only if $L(G)$ is k-connected or complete.

Let G be an essentially 2-edge-connected graph with $\overline{\sigma}_2(G) \ge 5$. Then $D(G) = D_1(G) \cup D_2(G)$ is an independent set. Let E_1 be the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 | x \in D_2(G)\}$. Define

$$G_0 = G/(E_1 \cup X_2(G)).$$

In other words, G_0 is obtained from G by deleting the vertices in $D_1(G)$ and replacing each path of length 2 whose internal vertex is a vertex in $D_2(G)$ by an edge. Note that G_0 may not be simple.

A vertex set $V(G_0)$ is regarded as a subset of $V(G)$. A vertex in G_0 is nontrivial if it is obtained by contracting some edges in $E_1 \cup X_2(G)$ or it is adjacent to a vertex in $D_2(G)$ in G. Since $\overline{\sigma}_2(G) \ge 5$, all vertex in $D_2(G_0)$ are nontrivial. Let $X = D(G)$. In [12], G_0 is denoted by $I_X(G)$. Following [10], we call G_0 the *core* of G.

Let G_0' be the reduction of G_0. For a vertex $v \in V(G_0')$. Let $\Gamma_0(v)$ be the maximum collapsible preimage of v in G_0 and let $\Gamma(v)$ be the preimage of v in G. Note that $\Gamma(v)$ is the graph induced by edges composing of $E(\Gamma_0(v))$ and some edges in $E_1 \cup X_2(G)$. A vertex v in G_0' is a *nontrivial vertex* if v is a contracted vertex (i.e., $|V(\Gamma(v))| > 1$) or v is adjacent to a vertex in $D_2(G)$.

In this paper, we improve Theorems 3 and 4 and get the following results. Where the graphs $G_1, G_2, \ldots, G_6$ are depicted in Fig. 2.

Theorem 5. *Let G be a connected almost bridgeless simple graph of order n such that*

$$\overline{\sigma}_2(G) \ge 2(\lfloor n/11 \rfloor - 1). \tag{1.2}$$

If n is sufficiently large, then either L(G) is traceable or $G_0' \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$, where G_0' is the reduction of G_0, G_0 is the core of G. Particularly, if $G_0' \in \{G_1, G_2, \ldots, G_6\}$, then G_0' is the reduction of G.

By Theorem 5, the following result follows immediately.

Corollary 6. *Let G be a connected almost bridgeless simple graph of order n with*

$$\delta(G) \ge \lfloor n/11 \rfloor - 1.$$