Applied Mathematics and Computation 321 (2018) 498–511

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stability analysis and numerical simulations of a one dimensional open channel hydraulic system

Boumediène Chentouf*, Nejib Smaoui

Kuwait University, Faculty of Science, Department of Mathematics, Safat 13060, Kuwait

ARTICLE INFO

Keywords:
Open channel hydraulic system
Output boundary feedback control
Stability
Chebychev collocation method

ABSTRACT

This paper is dedicated to the qualitative analysis as well as numerical simulations of a one dimensional open channel hydraulics system which is commonly used in hydraulic engineering to model the unsteady flow dynamics in a river. First, an output feedback control is proposed. Next, the closed-loop system is proved to possess a unique solution in a functional space. Furthermore, the spectrum and resolvent sets of the system operator are characterized. Then, stability results are stated and proved according to a smallness assumption on the feedback gain. The proof invokes Lyapunov direct method. Last but not least, we adopt the Chebychev collocation method, that uses backward Euler method and the Gauss-Lobatto points, to provide numerical simulations in order to ascertain the correctness of the theoretical outcomes.

1. Introduction

The flow dynamics in an irrigation canal are frequently modeled by complex partial differential equations. Notwithstanding, one of the most used and well admitted in literature is described via nonlinear coupled hyperbolic partial differential equations and called de Saint-Venant equations [20]:

$$\frac{\partial U}{\partial t} + \frac{\partial q}{\partial x} = l_d, \tag{1.1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{U}\right) + g\,U\frac{\partial H}{\partial x} = -g\,US_f + kVl_d, \tag{1.2}$$

in which x is the spatial location (m), t is the time (s), H is the absolute water surface elevation (m), $U(x, H)$ is the wetted cross-sectional area (m^2), $q(x, t)$ is the flow discharge (m^3/s), $S_f(q, H, x)$ the friction slope, $l_d(x, t)$ is the lateral discharge (m^2/s), that is, $l_d > 0$ represents an inflow, and $l_d < 0$ is the outflow. Furthermore, $V(x, t)$ is the mean velocity (m/s) in section U, and g the gravitational acceleration (m/s^2). Lastly, $k = 0$ if $l_d > 0$ and $k = 1$ if $l_d < 0$.

It is worth mentioning that (1.1) reflects the conservation of mass, while (1.2) is the conservation of momentum. In turn, it has been noticed that based on the characteristics of the river canal, the above model may have a simpler form using the following assumptions (for further discussion on the model, the reader is referred to [9,15,19]):

* Corresponding author.
E-mail addresses: chenboum@hotmail.com, boumediene.chentouf@ku.edu.kw (B. Chentouf), nsmaoui64@yahoo.com, n.smaoui@ku.edu.kw (N. Smaoui).

https://doi.org/10.1016/j.amc.2017.10.058

(i) the inertia terms $\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(\frac{q^2}{U})$ are negligible with respect to $g\,U\frac{\partial H}{\partial x}$;
(ii) the lateral inflow is minimal;
(iii) the flow variations as well as the bed slope of the river are small.

Indeed, the assumptions (i)–(iii) lead us to claim that the system (1.1) and (1.2) becomes

$$B\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{1.3}$$

$$\frac{\partial H}{\partial x} = -S_f, \tag{1.4}$$

where B is the water surface width. Next, a simple differentiation of (1.3) with respect to x and (1.4) with respect to t gives (see [15,19] for details)

$$B\frac{\partial S_f}{\partial t} = \frac{\partial B}{\partial x}\frac{\partial H}{\partial t} + \frac{\partial^2 q}{\partial x^2},$$

which together with (1.3) imply that

$$B\frac{\partial S_f}{\partial t} = -\frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial q}{\partial x} + \frac{\partial^2 q}{\partial x^2}. \tag{1.5}$$

Now, bearing in mind that the notation $(\frac{\partial f}{\partial u}(u,v))_v$ represents the variation of f with respect to u when v is fixed, we have

$$\left(\frac{\partial B}{\partial x}\right)_t = \left(\frac{\partial B}{\partial H}\right)_x \frac{\partial H}{\partial x} + \left(\frac{\partial B}{\partial x}\right)_H = -S_f\left(\frac{\partial B}{\partial H}\right)_x + \left(\frac{\partial B}{\partial x}\right)_H,$$

and

$$\left(\frac{\partial S_f}{\partial t}\right) = \frac{\partial S_f}{\partial q}\left(\frac{\partial q}{\partial t}\right)_x + \frac{\partial S_f}{\partial H}\left(\frac{\partial H}{\partial t}\right)_x = \frac{\partial S_f}{\partial q}\left(\frac{\partial q}{\partial t}\right)_x - \frac{1}{B}\frac{\partial S_f}{\partial H}\frac{\partial q}{\partial x}.$$

Whereupon, inserting the last two identities in (1.5) yields

$$\frac{\partial q}{\partial t} = \alpha(q,H,x)\,\frac{\partial^2 q}{\partial x^2} - \beta(q,H,x)\,\frac{\partial q}{\partial x},$$

where

$$\alpha(q,H,x) = \frac{1}{B\frac{\partial S_f}{\partial q}},$$

and

$$\beta(q,H,x) = \frac{1}{B\frac{\partial S_f}{\partial q}}\left(\frac{\partial S_f}{\partial H} - \frac{1}{B}\left[\frac{\partial B}{\partial H}(-S_f) + \frac{\partial B}{\partial x}\right]\right) = \frac{1}{B^2\frac{\partial S_f}{\partial q}}\left(\frac{\partial (BS_f)}{\partial H} - \frac{\partial B}{\partial x}\right).$$

Lastly, linearizing the above equation around a reference flow q_0, we obtain the desired diffusive wave equation:

$$\frac{\partial q_e(x,t)}{\partial t} = \alpha\,\frac{\partial^2 q_e(x,t)}{\partial x^2} - \beta\,\frac{\partial q_e(x,t)}{\partial x}. \tag{1.6}$$

Here, $q(x,t) = q_0 + q_e(x,t)$ and $q_e(x,\,t)$ is the deviation from the nominal flow q_0, whereas the positive constants α and β are respectively the diffusion and the celerity.

The above model is sometimes known as Hayami model described by a one-dimensional diffusive wave equation (see [9,19]).

Several techniques and methods have been proposed and utilized in order to provide efficient strategies for management of water distribution systems described by (1.6). For instance, one strategy consists in controlling water flow and this is the reason why the above equation with appropriate boundary and initial conditions has been the subject of an active numerical analysis by means of several discretization schemes (see [13–19]. In a series of recent articles, the authors in [2,6,7] focused on the qualitative analysis of the same equation under the presence of a proportional and/or integral controller but without approximating the system, that is, by using the infinite-dimensional systems theory [3,10]. In fact, the boundary integral control

$$\begin{cases} u(t) = K_I\xi(t), \\ \dot{\xi}(t) = y(t) - y_r, \end{cases} \tag{1.7}$$