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Linear multistep methods for impulsive delay differential equations☆

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ABSTRACT

This paper deals with the convergence and stability of linear multistep methods for a class of linear impulsive delay differential equations. Numerical experiments show that the Simpson's Rule and two-step BDF method are of order $p = 0$ when applied to impulsive delay differential equations. An improved linear multistep numerical process is proposed. Convergence and stability conditions of the numerical solutions are given in the paper. Numerical experiments are given in the end to illustrate the conclusion.

1. Introduction

Impulsive differential equations are widely used in actual modeling such as in ecology [27], population dynamic ([21,23]), optimal control [22] and so on. The studies of impulsive differential equations initiated in [18,19]. Since then, many results on the stability of impulsive differential equations have been studied (see [1,4,5,10,13]). In [20], Ran proved explicit Euler method is stable for impulsive differential equations while implicit Euler method is not stable. The surprising conclusion have attracted many scientist's concern on numerical properties of impulsive differential equations. The stability of Runge–Kutta methods for impulsive differential equations have been studied in [6,12,15,26]. For more difficult case, differential equations with non-fixed times of impulses are studied in [2,8]. But there is few papers focus on multistep methods since we showed classical linear multistep methods maybe not convergent for impulsive differential equations in [16]. In this paper, we will construct a convergent numerical scheme of linear multistep methods for impulsive delay differential equations and study the numerical stability.

In this paper, we consider the following equation:

$$x'(t) = ax(t) + bx(t-\tau), \; t > 0, \; t \neq d\tau, \tag{1.1a}$$

$$\Delta x = rx, \quad t = d\tau, \quad d = 0, 1, 2, \ldots, \tag{1.1b}$$

$$x(t) = \varphi(t), -\tau \leq t < 0, \tag{1.1c}$$

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where $a,\ b\in\mathbb{C},\ r\neq -1$ and φ is a smooth function on $[-\tau, 0)$ with $\lim_{t\to 0^-}\varphi(t)$ exists. Here we define $x(0)=(1+r)\lim_{t\to 0^-}\varphi(t)$.

Definition 1.1 (see [4]). $x(t)$ is said to be a solution of Eq. (1.1), if

1. $x(0)=(1+r)\lim_{t\to 0^-}\varphi(t)$,
2. $x(t)$ is differentiable and $x'(t)=ax(t)+bx(t-\tau)$ for $t\in(0,+\infty), t\neq d, d\in N$,
3. $x(t)$ is left continuous in $(0,+\infty)$ and $x(d^+)=(1+r)x(d), d\in N$.

2. Classical Linear multistep methods for impulsive delay differential equations

2.1. Linear multistep methods for ODEs

The standard form of classical linear multistep methods can be defined by

$$\sum_{i=0}^{k}\alpha_i x_{n+i-k}=h\sum_{i=0}^{k}\beta_i f_{n+i-k}, \tag{2.1}$$

where α_i and β_i are constants subject to the conditions

$$\alpha_k=1,\quad |\alpha_0|+|\beta_0|\neq 0, \tag{2.2}$$

and $f_{n+i-k}=f(t_{n+i-k},x_{n+i-k}),\ i=0,1,\ldots,k$.

2.2. Linear multistep methods for impulsive delay differential equations

Let $h=\frac{\tau}{m}$ be a given stepsize with integer m. In this subsection, we consider the case when $m\geq k$. The application of the linear multistep methods (2.1) in case of Eq. (1.1) yields,

$$x_{-m,\ i}=\varphi((i-m)h),\quad i=0,\ldots,m, \tag{2.3a}$$

$$x_{0,0}=(1+r)x_{-m,m}, \tag{2.3b}$$

$$x_{wm,\ l}=-\sum_{i=0}^{k-1}\frac{\alpha_i-ha\beta_i}{\alpha_k-ha\beta_k}x_{wm,\ l+i-k}+\sum_{i=0}^{k}\frac{hb\beta_i}{\alpha_k-ha\beta_k}x_{(w-1)m,\ l+i-k},\quad w=0,1,\ldots,\ \ l=k,\ldots,m, \tag{2.3c}$$

$$x_{(w+1)m,\ 0}=(1+r)x_{wm,\ m}, \tag{2.3d}$$

$$\begin{aligned}x_{(w+1)m,\ l}=&-\sum_{i=0}^{k-l-1}\frac{\alpha_i-ha\beta_i}{\alpha_k-ha\beta_k}x_{wm,\ m+l+i-k}-\sum_{i=k-l}^{k-1}\frac{\alpha_i-ha\beta_i}{\alpha_k-ha\beta_k}x_{(w+1)m,\ l+i-k}\\&+\sum_{i=0}^{k-l-1}\frac{hb\beta_i}{\alpha_k-ha\beta_k}x_{(w-1)m,\ m+l+i-k}+\sum_{i=k-l}^{k}\frac{hb\beta_i}{\alpha_k-ha\beta_k}x_{wm,\ l+i-k},\quad l=1,\ldots,k-1,\end{aligned} \tag{2.3e}$$

where $x_{wm,\ l}$ is an approximation of $x(t_{wm+l})$ and $x_{wm,\ m}$ denotes an approximation of $x((w+1)^-)$. Here we assume that the other starting value besides x_0, i.e., $x_{0,1},\ldots,x_{0,k-1}$, have been calculated by an one-step method.

Remark 2.1. As a special case, when $k=1$ the corresponding consistent process (2.3) takes the form:

$$x_{-m,\ i}=\varphi((i-m)h),\quad i=0,\ldots,m, \tag{2.4a}$$

$$x_{0,0}=(1+r)x_{-m,m}, \tag{2.4b}$$

$$\begin{aligned}x_{wm,\ l}=&\left(1+\frac{ha}{1-ha\beta_1}\right)x_{wm,\ l-1}+\frac{hb\beta_0}{1-ha\beta_1}x_{(w-1)m,\ l-1}+\frac{hb\beta_1}{1-ha\beta_1}x_{(w-1)m,\ l},\\&l=1,\ldots,m,\ \ w=0,1,\ldots,\end{aligned} \tag{2.4c}$$

$$x_{(w+1)m,\ 0}=(1+r)x_{wm,\ m}, \tag{2.4d}$$