Applied Mathematics and Computation 321 (2018) 577–592

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Linear-quadratic partially observed forward–backward stochastic differential games and its application in finance☆

Zhen Wu, Yi Zhuang*

School of Mathematics, Shandong University, Jinan, China

ARTICLE INFO

MSC:
91A23
60H10
93E11
91G80

Keywords:
Linear-quadratic problem
Stochastic differential game
Forward–backward stochastic differential equation
Partial information
g-expectation

ABSTRACT

This paper is concerned with a partially observed linear-quadratic game problem driven by forward–backward stochastic differential equations where the forward diffusion coefficients do not contain control variables and the control domains are not necessarily convex. The drift term of the observation equation is linear with respect to the state, and there is correlated noise between the state and the observation equation. By virtue of the classical spike variational method and the backward separation technique, we derive a necessary and a sufficient condition of the stochastic differential game problem. Then we obtain filtering equations and present a feedback representation form of the equilibrium point through Riccati equations. As a practical application, we solve a partial information investment problem involving g-expectation as a convex risk measurement and give the numerical simulation to show the explicit investment strategy and illustrate some reasonable phenomena influenced by common financial parameters.

1. Introduction

The general theory of the backward stochastic differential equation (BSDE) was first introduced by Pardoux and Peng [1]. If a BSDE coupled with a forward stochastic differential equation (SDE), it is called the forward–backward stochastic differential equation (FBSDE). In the stochastic control area, the form of the classical Hamiltonian system is one kind of FBSDEs. The classical Black–Scholes option pricing formula in the financial market can be deduced by a certain FBSDE. Some fundamental research based on FBSDEs is surveyed by Ma and Yong [2].

The game theory was first introduced by Von Neumann and Morgenstern [3]. Nash [4] made the fundamental contribution in Non-cooperate Games and gave the notion of Nash equilibrium point. In recent years, many articles on stochastic differential game problems driven by stochastic differential equations appeared. Researchers try to consider the influence on several players rather than one player and try to find an equilibrium point rather than an optimal control. These problems are more complicated than the classical control problems but much closer to the reality. Yu and Ji [5] studied the LQ backward case. Yu [6] solved the LQ case on forward and backward system. Øksendal and Sulem [7], Hui and Xiao [8] made a research on the maximum principle of forward–backward system.

☆ This work was supported by the Natural Science Foundation of China (61573217), the National High-level personnel of special support program of China and the Chang Jiang Scholar Program of Chinese Education Ministry.

* Corresponding author.
E-mail addresses: wuzhen@sdu.edu.cn (Z. Wu), kevinchrist@126.com (Y. Zhuang).

https://doi.org/10.1016/j.amc.2017.11.015
0096-3003/

Throughout the paper, we denote by $\mathbb{R}$ the Euclidean space; by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ the complete probability space. For the classical stochastic differential game system, we have the following stochastic differential equation (SDE):

$$\begin{cases} dx^v(t) = b(t, x^v(t), v_1(t), v_2(t))dt + \sigma(t, x^v(t), v_1(t), v_2(t))dW(t), \\ x^v(0) = x_0, \end{cases}$$

where $v(\cdot) = (v_1(\cdot), v_2(\cdot))$, $W(\cdot)$ is a Brownian motion, $\mathcal{F}_t = \sigma\{W(s)|0 \le s \le t\}$ is the natural filtration.

The cost functional:

$$J(v_1(\cdot), v_2(\cdot)) = \mathbb{E}\left[\int_0^T f\big(t, x^v(t), v_1(t), v_2(t)\big)dt + h(x^v(T))\right]. \tag{1.1}$$

Under some well-posed assumption the system above, the general zero-sum stochastic differential game problem is represented as follows:

Problem $\mathbf{P_0}$. For any $x_0 \in \mathbb{R}$, find a pair of $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J(u_1(\cdot), u_2(\cdot)) = \sup_{v_2(\cdot)\in\mathcal{U}_2}\left(\inf_{v_1(\cdot)\in\mathcal{U}_1} J(v_1(\cdot), v_2(\cdot))\right) = \inf_{v_1(\cdot)\in\mathcal{U}_1}\left(\sup_{v_2(\cdot)\in\mathcal{U}_2} J(v_1(\cdot), v_2(\cdot))\right).$$

where $\mathcal{U}_1 \times \mathcal{U}_2$ is a certain admissible control set.

However, for the *Problem P_0*, the expectation in (1.1) is just a linear expectation that cannot always represent the practical situation in the real world. In behavioral economics, such non-linear phenomenon can be regarded as personal preference or risk measurement in many articles including in [9–11]. Here, we replace the linear expectation by a generalized expectation called g-expectation introduced by Peng [12,13], which can be seen as a *convex risk measure* and is induced by a backward stochastic differential equation (BSDE).

Consider the following BSDE:

$$\begin{cases} -d\eta(t) = g(t, \zeta(t))dt - \zeta(t)dW(t), \\ \eta(T) = \xi. \end{cases} \tag{1.2}$$

Under certain assumptions, (1.2) exists a unique solution $(\eta(\cdot), \zeta(\cdot))$. If we also set $g(\cdot, 0) \equiv 0$, we can make the definition as follows:

Definition 1. For each $\xi \in \mathcal{F}_T$, we call

$$\mathcal{E}_g(\xi) \triangleq \eta(0), \tag{1.3}$$

the generalized expectation (g-expectation) of ξ related to g.

We can know that the map $\xi \to \mathcal{E}_g(\xi)$ includes all the properties that $\mathbb{E}$ have, except the linearity. And it is obvious that when $g(\cdot) = 0$, $\mathcal{E}_g$ is reduced to the classical expectation $\mathbb{E}$.

Thus we can define the following cost functional with g-expectation:

$$J_g(v_1(\cdot), v_2(\cdot)) = \mathcal{E}_g\left[\int_0^T f\big(t, x^v(t), v_1(t), v_2(t)\big)dt + h(x^v(T))\right]. \tag{1.4}$$

And our problem can be formulated in a generalized case as follows:

Problem $\mathbf{P_g}$. For any $x_0 \in \mathbb{R}$, find a pair of $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J_g(u_1(\cdot), u_2(\cdot)) = \sup_{v_2(\cdot)\in\mathcal{U}_2}\left(\inf_{v_1(\cdot)\in\mathcal{U}_1} J_g(v_1(\cdot), v_2(\cdot))\right) = \inf_{v_1(\cdot)\in\mathcal{U}_1}\left(\sup_{v_2(\cdot)\in\mathcal{U}_2} J_g(v_1(\cdot), v_2(\cdot))\right),$$

where $\mathcal{U}_1 \times \mathcal{U}_2$ is a certain admissible control set.

From (1.2)–(1.4), we can see that

$$\eta(T) = \xi(v_1(\cdot), v_2(\cdot)) = \int_0^T f\big(t, x^v(t), v_1(t), v_2(t)\big)dt + h(x^v(T)).$$

Now we define

$$\begin{cases} y^v(t) = \eta(t) - \int_0^t f(s, x^v(s), v_1(s), v_2(s))ds, \\ z^v(t) = \zeta(t), \end{cases}$$

then we have the following BSDE:

$$\begin{cases} -dy^v(t) = (g(t, z^v(t)) + f(t, x^v(t), v_1(t), v_2(t)))dt - z^v(t)dW(t), \\ y^v(T) = h(x^v(T)), \end{cases} \tag{1.5}$$

where $(y(\cdot), z(\cdot))$ is the unique adapted solution.