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Using energy methods to compare linear vibration models

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ABSTRACT

In this paper, it is proved that the series representation of the solutions of the general linear vibration model with modal damping, is valid with respect to the energy norm. The partial sums of the series representation of solutions may therefore be used to compare different models. Parseval-type relationships are obtained for the eigenfunction expansions of functions. These relationships are used to derive expressions for the relative error in energy for partial sum approximations of functions. An example is included where a string model (wave equation model) and a modified Euler–Bernoulli beam model for the transverse vibration of a steel wire are compared.

1. Introduction

A variety of linear vibration models (one, two or three dimensional), derived from linear elasticity theory, are available. Often more than one model can be applied to a specific situation. It then becomes important to compare the different models and decide on the validity ranges of the models. For example, in recent papers the Timoshenko beam model is compared to higher dimensional models; see [1] and [2]. Also, in [2] and [3] the Euler–Bernoulli, Timoshenko and other beam models are compared. In all these cases the authors use the natural frequencies as a basis for the comparison of the models. The underlying assumption seems to be that as solutions obtained by the method of separation of variables, are expressed in terms of the natural frequencies and modes, these solutions can then also be compared. As mentioned in the conclusion of [2], "...what is really required is to determine the difference between solutions that result from the same disturbance." It is not a trivial matter to compare these solutions, since they are not known explicitly. In this paper we establish a mathematical foundation for such a comparison: use approximate solutions with rigorously determined error estimates.

There are also countless examples where mathematical models are simplified by making additional assumptions, e.g. neglecting terms which are considered to be small. But there are few examples where solutions are compared to see if the additional assumptions are actually justified.

We aim to devise a practical procedure based on rigorous mathematics. In order to formulate linear vibration problems in an abstract setting, we consider Hilbert spaces V and W. We assume that V is a dense subset of W. Symmetric bilinear forms a and b are defined on V and the symmetric bilinear form c is defined on W. (See Section 2 for details.)

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The *general linear vibration problem* is to find a function u such that for each $t > 0$, the value $u(t)$ is in V, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_W \text{ for each } v \in V.$$

In addition, u satisfies the initial conditions $u(0) = u_0$ and $u'(0) = u_1$.

In applications, the spaces V and W are Sobolev spaces or subspaces of Sobolev spaces.

In Section 2, we consider sufficient conditions for the existence of solutions of the problem above. In Sections 3–5, various aspects of the validity of the series representation of these solutions are discussed. Section 6 starts with an explanation of how energy principles, combined with the series representation of solutions, can be used to compare different models. As an application, two different models for the transverse vibration of a long thin beam (e.g. a steel piano wire), are presented. Section 7 contains a comparison of the eigenfrequencies and eigenfunctions of the two models and in Section 8 we show how solutions resulting from the same initial disturbance can be compared. Some concluding remarks are given in Section 9.

This paper is based on the unpublished technical reports [4] and [5].

2. Existence

In this section, we present an existence theorem for a linear vibration problem with modal damping. This result is a special case of a general existence theorem in [6]. For this theorem the assumptions are given in terms of the bilinear forms a, b and c, which is convenient. In order to verify these assumptions, it is not necessary to construct linear operators with suitable properties as required by the existence results found in other publications, e.g. [7–11].

Free vibration problem: Problem G

Find u such that for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = 0 \quad \text{for each } v \in V, \tag{1}$$

satisfying $u(0) = u_0$ and $u'(0) = u_1$.

Assumptions. We assume that the following properties hold:

- **A1** W is a Hilbert space with inner product c and induced norm $\|\cdot\|_W$.
- **A2** V is a Hilbert space with inner product b and induced norm $\|\cdot\|_V$.
- **A3** There exists a constant k_b such that $\|v\|_W \le k_b \|v\|_V$ for each $v \in V$.
- **A4** V is a dense subset of W.

The spaces W and V are referred to as the *inertia space* and the *energy space*, respectively.

We consider the case of *modal damping*. That means that the bilinear form a is given by $a = \alpha b + \beta c$, with α and β nonnegative constants, and one or both can be zero. Note that *viscous type damping* ($\alpha = 0$) and *no damping* ($\alpha = \beta = 0$) are included as special cases of modal damping. The restriction to modal damping is used as no general spectral theory is available for the case of non-modal damping. Clearly, in the case of modal damping, a is nonnegative, symmetric and bounded on V, and the following existence theorem, proved in [6], applies.

Theorem 2.1. *Suppose Assumptions A1–A4 hold. If for $u_0 \in V$ and $u_1 \in V$, there exists some $y \in W$ such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \text{ for each } v \in V, \tag{2}$$

then there exists a unique solution $u \in C^1([0, \infty), V) \cap C^2([0, \infty), W)$ for Problem G.

Note that $u'(t) \in X$ implies that the derivative is defined in terms of the norm of the space X, and that $u \in C^m(I, X)$ for some interval I implies that u and its derivatives up to order m are continuous on I.

For modal damping the condition (2) has an alternative formulation in terms of the equilibrium states E_b.

Definition. Let E_b be the subset of V defined as follows: $x \in E_b$ if there exists a $y \in W$ such that

$$b(x, v) = c(y, v) \quad \textit{for each } v \in V. \tag{3}$$

In the case of modal damping, for $u_0 \in V$ and $u_1 \in V$, condition (2) is equivalent to $u_0 + \alpha u_1 \in E_b$.

Also, if $u_0 \in E_b$ and $u_1 \in E_b$, then it is clear that (2) holds.

The method of separation of variables may be used in the case of modal damping. The vibration problem (1) has a solution of the form $u(t) = T(t)w$ if and only if

$$b(w, v) = \lambda c(w, v) \textit{ for each } v \in V \tag{4}$$

and

$$T''(t) + (\beta + \lambda\alpha)T'(t) + \lambda T(t) = 0. \tag{5}$$

In Section 4, it is proved that there exists an orthogonal sequence of eigenvectors $\{w_n\}$ for eigenvalue problem (4) with a corresponding sequence of positive eigenvalues $\{\lambda_n\}$.