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Peristaltic axisymmetric flow of a Bingham fluid

L. Fusi*, A. Farina

Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Viale Morgagni 67/a, Firenze 50134, Italy

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A B S T R A C T

We model the peristaltic flow of a Bingham fluid in a tube in lubrication approximation. Following the procedure developed in Fusi et al. (2015a) we derive the rigid plug equation using an integral formulation for the balance of linear momentum, modelling the unyielded domain as an evolving non-material volume. The mathematical problem is formulated for the yielded and unyielded part and appropriate boundary conditions are established at the pipe walls and at the yield surface. The zero order approximation leads to a system formed by an integral equation and an algebraic equation for the yield surface and for the plug velocity (which is uniform in space), respectively. Because of the integral approach adopted in the unyielded part of the flow, the leading order approximation does not give rise to the lubrication paradox. The problem is solved numerically and an analytical solution is found when the oscillating wall is given as a small perturbation of the uniform wall.

1. Introduction

Peristaltic flows in channels or ducts are generated by the continuous periodic contraction and expansion of the flexible walls. The mechanism of peristalsis occurs in different branches of biomechanics and it is of prime importance when considering, for instance, how physiological fluids such as blood and urine are transported in the human body. Among the many bio-mechanical processes involving peristalsis we find the swallowing of food through the oesophagus, the movement of chyme through the intestine, spermatic flow in the male reproductive system, the movement of eggs in the fallopian tube and the transport of bile. Other non-biological important applications are the design of finger and roller pumps used in pumping fluids where contamination due to contact with the pumping machinery has to be avoided, the transport of highly viscous fluids or slurries, the design of dialisys machines, open-heart bypass, infusion pumps etc.

For all its potential applications peristalsis has been the subject of intensive theoretical and experimental studies in the past, see [2,7,11,12]. When modelling peristaltic motion, the main scope is to characterize the basic fluid mechanics of the process and to determine the velocities and the pressure gradients that are generated by the wave. To obtain analytical solutions the models often rely on simplifying assumptions such as vanishingly small Reynolds number (creeping flow), infinitely long wavelength, small wave amplitude etc. Since the type of fluid commonly involved in peristaltic motion is non-Newtonian, a large part of the theoretical modelling is focussed on non-Newtonian fluids such as Jeffrey fluid, Casson fluid, Herschel-Bulkley fluid, Bingham fluid, Power-law fluid, etc. We refer the reader to the paper [6], where a schematic summary of key assumptions made by recent literatures related to peristaltic transport is available.

* Corresponding author.
E-mail addresses: lorenzo.fusi@unifi.it, lorefusi@gmail.com (L. Fusi).

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In this paper we investigate the peristaltic motion of a Bingham fluid flowing in a cylindrical deformable tube whose shape can be described by a periodic travelling wave. In particular we consider a flow which occurs in lubrication regime, that is we assume that the characteristic length of the tube is far larger than the typical radius. Although this kind of study is certainly not new (see for instance [8–10]), here we investigate the motion adopting a novel approach which consists in deriving the plug momentum equation using an integral formulation. This methodology has been introduced in [3] to model the planar flow of a Bingham plastic in a channel of varying width. The key feature is to treat the unyielded phase as an evolving non-material volume and to write the momentum balance globally and not locally. Indeed, it is well known that when modelling the plug with the classical differential formulation

$$\rho^* \frac{D\mathbf{v}^*}{Dt^*} = \nabla^* \cdot \boldsymbol{\sigma}^* \tag{1}$$

that requires the knowledge of the stress components σ^*_{ij} in the unyielded domain, we may end up with the so-called *lubrication* paradox, which consists in getting a plug velocity which depends on the longitudinal coordinate of the tube (see [1]). In [3] we have proved that this paradox arises because one cannot model the unyielded part using the stress components which are not even defined for that part of the fluid. To avoid this inconsistency we have treated the rigid part of the flow as an evolving non-material volume and we have written the momentum equation in the following integral form

$$\int_{\Omega^*_u} \frac{\partial}{\partial t^*}(\rho^* \mathbf{v}^*) dV^* + \int_{\partial\Omega^*_u} \rho^* \mathbf{v}^* (\mathbf{v}^* \cdot \mathbf{n}) dS = \int_{\partial\Omega^*_u} \boldsymbol{\sigma}^* \mathbf{n} dS, \tag{2}$$

where it is only required to know the stress $\boldsymbol{\sigma}^*$ exerted by the yielded part at the yield surface. Such a stress can be evaluated solving Eq. (1) in the yielded part. Using this strategy in [3] we have shown that it is possible to find an explicit expression for the yield surface and to determine the inner plug that moves with uniform velocity even for non-uniform channel walls. This method has been subsequently used to study the planar squeeze flow of a Bingham fluid [4] and the flow of Bingham fluid down an inclined channel [5]. Here we use the same procedure to model the peristaltic flow of a Bingham fluid in a tube where the walls evolve as a travelling wave.

The paper is organized as follows. We derive the model for a generic 3D setting using cylindrical coordinates (r^*, θ, z^*) and assuming that the main variables of the problem do not depend on θ. We rescale the problem assuming that the aspect ratio ε – corresponding to the ratio between the characteristic length of the tube and the characteristic radius – is small. Then we focus on the leading order approximation and we show that the mathematical problem reduces to a set of two equations (one of which is an integral equation) involving the velocity of the plug and the yield surface. Of course the general problem can be solved only numerically. Finally we assume that the oscillation amplitude of the wall is a *small parameter* too and we show that setting this parameter to zero corresponds to considering the classical Bingham model in cylindrical geometry with fixed walls. Finally we show that assuming that the oscillating wall is a small perturbation of the fixed wall, an analytical solution can be found as a *small perturbation* of the solution with uniform wall. For this latter case we plot the evolution of the yield surface, the plug velocity and the pressure gradient at different times. Moreover we study the dependence of the solution on the principal parameters of the model, i.e., the Bingham number and the prescribed inlet discharge.

2. Mathematical formulation of the problem

We consider the flow of a Bingham fluid in a pipe of circular-cross section whose amplitude[1] R^* evolves as a traveling wave. We suppose that the length of the pipe is L^*. The flow is modelled considering cylindrical coordinates (r^*, θ, z^*) and assuming that all the kinematical quantities appearing in the system do not depend on θ. The velocity field is of the form

$$\mathbf{v}^*(r^*, z^*, t^*) = v^*_r(r^*, z^*, t^*)\mathbf{e}_r + v^*_z(r^*, z^*, t^*)\mathbf{e}_z$$

Referring to Fig. 1 we assume that the travelling wave describing the motion of the wall is given by

$$R^*(z^*, t^*) = R^*_{ref}\left[1 + \delta\varphi\left(\frac{z^*}{\lambda^*} - \frac{t^*}{T^*}\right)\right] \tag{3}$$

where $R^*_{ref}\delta$ is semi-amplitude of the wave with $\delta \in (0, 1)$, $\varphi \in [-1, 1]$ is a smooth periodic function of period one, λ^* is the wavelength and T^* is the wave period. The stress $\boldsymbol{\sigma}^*$ can be decomposed in the following form

$$\boldsymbol{\sigma}^* = -p^*\boldsymbol{I} + \boldsymbol{\tau}^*,$$

where $\boldsymbol{\tau}^*$ is traceless deviatoric part of the stress and $p^* = -(1/3)\,\mathrm{tr}\,\boldsymbol{\tau}^*$ is the mean normal stress, or mechanical pressure.

The Bingham stresses τ^*_{ij} are related to the strain rates $\dot{\gamma}^*_{ij}$ through the constitutive equations

$$\begin{cases} \tau^*_{ij} = \left(2\mu^* + \dfrac{\tau^*_o}{\dot{\gamma}^*}\right)\dot{\gamma}^*_{ij} & \tau^* \geqslant \tau^*_o, \\[2ex] \gamma^*_{ij} = 0 & \tau^* \leqslant \tau^*_o, \end{cases}$$

[1] The starred variables denote dimensional quantities.