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Nonlinear dynamics of discrete time multi-level leader–follower games

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A B S T R A C T

We study dynamic multiple-player multiple-level discrete time leader–follower games in the vein of Cournot or Stackelberg games; these games generalize two-player dynamic Stackelberg or Cournot duopoly games which have been considered recently. A given player acts as a leader toward players in lower levels, and as a follower toward players in higher levels. We consider the case of either perfect or incomplete information, which in this context means that players either have complete information about other players within their level (perfect information) or lack information at the present timestep about other players within their level (incomplete information). Players always have perfect information about all players which are (relative) followers, and incomplete information about players which are (relative) leaders. The Cournot-type adjustment process under these information structures at each timestep results in the temporal dynamics of the game. As we consider dynamic games, we observe a variety of behaviors in time, including convergence to steady state or equilibrium quantities, cycles or periodic oscillations, and chaotic dynamics. We find that the costs facing each player strongly influence the form of the long-time dynamics, as will the information structure (perfect or incomplete) selected. One interesting finding is that under perfect information players tend to quickly converge upon their respective equilibrium values, while incomplete information can result in loss of regularity and the emergence of periodic or chaotic dynamics. However, in cases where players may be pushed out of the game in the presence of high relative costs and perfect information, we find that non-equilibrium dynamics under incomplete information allow such players to retain positive production, hence they are able to remain in the game.

1. Introduction

There have been a number of studies on Cournot games for two (duopoloy) or more players, phrased as dynamic games on discrete or continuous time domains [1,2,4,10,24–27,33].

In Cournot games, all players have equal information and make moves simultaneously. However, real-life asymmetry in such dynamics has motivated the consideration of other models. In particular, one direction one can take is to consider leader–follower models, where a leader has more information on the state of the game than does a follower. A classical example of such a game would be a Stackelberg game [39] which is a type of leader–follower game. In the Stackelberg framework, two players no longer decide their outputs independently yet simultaneously. Instead, the leader decides their

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move first, given knowledge of the situation at the previous timestep [30]. When the leader knows the reaction function of the follower, which depends on the output of leader, then it can determine its own output by maximizing its profit function. After the leader's decision is implemented, the follower has to produce the amount that has already been determined by the leader due to the reaction function. Since in a leader–follower model the leader is able to obtain the response of the followers given any policy made by the leaders, the leader is able to obtain the best output and reach their equilibrium in one iteration. Dynamic Stackelberg games have been considered, with a variety of dynamics (including periodic and chaotic behaviors) observed; see [21,22,26,34]. It is possible to extend such models to multiple players which may take the role of either leaders or followers (see, for instance, [31,37]). In addition to classical economic applications, such dynamic leader–follower games have seen recent application in areas such as communications networks [40], signaling games [3], and service-for-prestige theory [23].

Stackelberg strategies were considered for multilevel systems by Cruz [6]. Such concepts are useful in the multilevel hierarchical control problem of multicontroller distributed-parameter systems [7,35]. [16] considered discrete-time dynamic multi-leader–follower games with the assumption of feedback perfect information, which extended the static multi-leader–follower games of [18]. Applications of multi-level or hierarchical games and related systems include internet pricing [29], information theory [8], incentives [14], and cellular network deployment [12].

While discrete and continuous time dynamics have been considered in the past 20 years or so for variations of the two- or many-player dynamic Cournot games [1,2,4,10,26,27,33], and dynamic Stackelberg games and related variations on leader–follower games [21,22,26,34], the corresponding nonlinear dynamics arising from multi-level leader–follower games has not been well-studied. In the present paper, we shall consider two-level and then multi-level leader–follower dynamic games in discrete time. As expected, the dynamics depend not only on the parameter values, but also on the information structure adopted, with a range of dynamics from stable equilibrium, to limit cycles and periodic orbits, and then chaos, being found. Indeed, there are interesting findings which do not seem to be studied in two player leader–follower games, or games with only two-tier hierarchies.

Game theory has of course seen application to economic problems such as Cournot adjustment processes involving competing firms. However, there are a variety of other applications for game theory in the dynamic setting which may be considered. Such research could be of relevance for effective vaccination strategies [41], for understanding human cooperation [20] and collusion [42] or more general human behavior [28], to combat crime [9], standards of quality for products or services [36], or even for saving human lives in general [13]. Hence, a better understanding of the dynamics of such games can be of benefit outside of economics, as well.

The paper is organized as follows. Section 2 reviews useful aspects of the dynamic Cournot games. In Section 3, we introduce generalizations of duopoly games to two-level (a leader level and a follower level), with each level having some positive integer number of players. We discuss the role of incomplete information versus perfect information within the leader level, and show that incomplete information may result in a loss of regularity and the emergence of non-equilibrium dynamics, such as periodic oscillations and chaos. In Section 4, we extend the model further, to consider multiple levels, with players in each level acting as leaders to those in levels below, and as followers to those in levels above. Again, we find that information structure influences the dynamics observed. Finally, we summarize and discuss the results in Section 5.

2. Review of Cournot model results

In this section we shall review material on Cournot dynamic games, as this will help to motivate the more general models which follow in Sections 3 and 4. The results here motivate the need to consider information asymmetry. For related differential games related to control theory, leader–follower games with multiple-players have been discussed in several contexts. Optimal control approaches for the dynamic games are discussed in [37] (and references therein). Multiple-level leader-follower games which are Stackelberg-like in nature have been studied by Cruz [7]. Gardner and Cruz [11] considered feedback Stackelberg games for the multi-level or hierarchical setting, while the closed-loop control was studied by Baser and Olsder [5]. A control problem for another class of hierarchical leader-follower games was considered by Pan and Yong [17].

Two frequently used functional forms for price in terms of quantity are $P = A - Q$ or $P = A/Q$ [34] where A is a constant and Q is the total output quantity. Sometimes, a variable can be added to represent government intervention such as taxation or subsidy [33]. We can also set a upper bound for the price to avoid infinite price as Q tends to zero for $P = \frac{1}{Q}$ [33]. Since we find that players with linear price functions always approach fixed equilibrium values in finite time, this paper focuses on the inversely proportional price function $P = A/Q$. This function has been employed in many of the aforementioned studies on Cournot or Stackelberg games, and hence maintains relevance to possible applications.

The Cournot model describes a market where all the players in the market are independent decision makers. Each player only knows the previous outputs of the other players, $q_j(t)$, and will try to maximize its profit at the next time step

$$\Pi_i(t+1) = q_i(t+1)P(\mathbf{q_{j:j\neq i}}(t), q_i(t+1)) - c_i q_i(t+1), \tag{1}$$

where P is the price function which depends on the total output, $q_i(t+1)$ is the output of the ith player at time $t+1$, $\mathbf{q_{j:j\neq i}}(t)$ represents the vector of outputs for all the other players, and c_i is the marginal cost faced by the ith player.

The profit (1) is maximized when $q_i(t+1)$ satisfies

$$P(\mathbf{q_{j:j\neq i}}(t), q_i(t+1)) + q_i(t+1)P'(\mathbf{q_{j:j\neq i}}(t), q_i(t+1)) - c_i = 0. \tag{2}$$