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The Nekrasov diagonally dominant degree on the Schur complement of Nekrasov matrices and its applications[☆]

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ABSTRACT

In this paper, we estimate the Nekrasov diagonally dominant degree on the Schur complement of Nekrasov matrices. As an application, we offer new bounds of the determinant for several special matrices, which improve the related results in certain case. Further, we give an estimation on the infinity norm bounds for the inverse of Schur complement of Nekrasov matrices. Finally, we introduce new methods called Schur-based super relaxation iteration (SSSOR) method and Schur-based conjugate gradient (SCG) method to solve the linear equation by reducing order. The numerical examples illustrate the effectiveness of the derived result.

1. Introduction and preliminaries

It is well known that Nekrasov matrices have wide applications in many practical problems such as linear system and control theory ([1,2]). In recent years, there are many researchers pay much attention to discussing the properties of Nekrasov matrices (see [2–9]). For instance, Pang et al. [2], Li [3] gave some properties of Nekrasov matrices. Kolotilina [4], Bailey and Crabtree [5] got several upper and lower bounds for the determinant of Nekrasov matrices. Gao et al. [6], Cvetković et al. [7–9] obtained infinity norm bound for the inverse of Nekrasov matrices.

In large scale linear systems, Schur complement is a useful tool in reducing the order. For example, consider the linear equation $Ax = b$, partitioning A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, where A_{11} is nonsingular. Then $Ax = b$ is equivalent to the pair of linear equations

$$A_{11}x_1 + A_{12}x_2 = b_1, \tag{1}$$

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$$A_{21}x_1 + A_{22}x_2 = b_2. \tag{2}$$

Multiplying (1) by $-A_{21}A_{11}^{-1}$ and adding it to (2), the vector variable x_1 is eliminated. Then we get a smaller size linear system as follows

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})x_2 = b_2 - A_{21}A_{11}^{-1}b_1. \tag{3}$$

For one thing, if the coefficient matrix A is strictly diagonally dominant, strictly doubly diagonally dominant and generalized doubly diagonally dominant, the coefficient matrix of smaller size linear system (3) is the same type matrix. Further, under certain conditions, the Schur complements of γ-diagonally and product γ-diagonally dominant matrices are the same type matrices as well. Hence, we could use Schur-based iteration methods to solve linear system (3), and these iteration methods are faster than the ordinary iteration methods (see [10–12]). However, a Nekrasov matrix may not be one of these matrices above. Thus, if the coefficient matrix A is a Nekrasov matrix, whether its Schur complement is the same type matrix and whether we could use the iteration methods in [10–12], even how to design the iterative scheme are important and unsolved issues.

For another, in order to design iteration methods, we need to estimate the spectral radius or the norm of the inverse for the coefficient matrix of (3). When it is smaller than 1, we could use Schur-based iteration methods. Further, the smaller of the norm for the inverse of the coefficient matrix, the faster of the Schur-based iteration methods (see [10–12]). Hence, estimating the Nekrasov diagonally dominant degree on the Schur complement of Nekrasov matrices and the norm of the inverse for the coefficient matrix of (3) are potential and unsolved issues as well.

Based on the above, studying the properties of Nekrasov matrices and designing Schur-based iteration method may have important theoretical and practical significance. In this paper, we first estimate the Nekrasov diagonally dominant degree on the Schur complement of Nekrasov matrices. Then, as an application, we present several bounds for the determinants of Nekrasov matrices and strictly (doubly) diagonally dominant matrices, which improve and extend the previous results. Finally, we provide an estimation about the infinity norm bounds for the inverse of Schur complement of Nekrasov matrices and introduce new methods called Schur-based super relaxation iteration (SSSOR) method and Schur-based conjugate gradient (SCG) method to solve the linear equation $Ax = b$ by reducing order, which can be used to compute out the solutions of $Ax = b$ faster than several known methods in some sense.

To begin with the main results, we need the following symbols and definitions. Let $\mathbb{C}^{n\times n}$ ($\mathbb{R}^{n\times n}$) denote the set of all $n \times n$ complex (real) matrices and $\mathbb{N} = \{1, 2, \ldots, n\}$. For $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, we denote by $A \geq (>) 0$ the nonnegative (positive) matrix, $|A| = (|a_{ij}|)$, $\rho(A)$ stands for the spectral radius of A, and

$$P_i(A) = \sum_{j\in N, j\neq i} |a_{ij}|, \ \ i \in \mathbb{N}.$$

Recall that $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ is called diagonally dominant (D_n) if

$$|a_{ii}| \geq P_i(A), \ \ i \in \mathbb{N}. \tag{4}$$

Further, A is said to be a strictly diagonally dominant matrix (SD_n) if all the inequalities in (4) are strict.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an M-matrix if it can be written as $A = sI_n - B$, where $B \geq 0$ and $s \geq \rho(B)$, I_n denotes a unit matrix of order n.

For $\alpha \neq \emptyset, \alpha^c \subset \mathbb{N}$, which are arranged in increasing order, we denote by $|\alpha|$ the cardinality of α and let $A(\alpha, \alpha^c)$ stand for the sub-matrix of $A \in \mathbb{C}^{n\times n}$ lying in the rows indicated by α and the columns indicated by α^c. In particular, $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. If $A(\alpha)$ is nonsingular, then the Schur complement of A with respect to $A(\alpha)$ is denoted by $A/A(\alpha)$ or simply A/α, to be

$$A/\alpha = A(\alpha^c) - A(\alpha^c, \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha^c).$$

Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $a_{ii} \neq 0$ $(i \in \mathbb{N})$, we define $R_i(A)$ recursively:

$$R_1(A) = \sum_{j=2}^{n} |a_{1j}|, \ \ R_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \ \ i \in \mathbb{N} - \{1\}.$$

Then A is called a Nekrasov matrix and denoted by $A \in N_n$ if

$$|a_{ii}| > R_i(A), \ \ i \in \mathbb{N}.$$

We call $|a_{ii}| - R_i(A)$ $(i \in \mathbb{N})$ the Nekrasov diagonally dominant degree of A with respect to the ith row.

Remark 1.1. Obviously, by computation, it is easy to see that if $A \in SD_n$, then $A \in N_n$.

2. The Schur complement of Nekrasov matrices

In this section, we estimate the Nekrasov diagonally dominant degree on base of the Schur complement of Nekrasov matrices. Note that the Schur complement of H-matrices are H-matrices (see [13]). However the Schur complement of Nekrasov matrices may not be Nekrasov matrices.