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# A nonlinear multigrid method for inverse problem in the multiphase porous media flow

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ABSTRACT

In this paper, we consider a parameter identification problem for the nonlinear convection–diffusion equation in the multiphase porous media flow. A nonlinear multigrid method is proposed for the recovery of permeability. This method works by dynamically adjusting the objective functionals at different grids so that they are consistent with each other, and ultimately reduce, the finest grid objective functional. In this manner, the nonlinear multigrid method can efficiently compute the solution to a desired fine grid inverse problem. Numerical results illustrate that the proposed multigrid approach both dramatically reduces the required computation and improves the reconstructed image quality.



## 1. Introduction

The multiphase flow in porous media has received considerable attention in recent years because of increasing applications in science and engineering. Applications of porous media flow include Biology [1], Mechanics [2], Geosciences [3], Civil Engineering [4]. In many engineering problems, such as those found in reservoir simulation, the oil and gas flow within the reservoir is often modeled as porous media flow. In [5], Espedal and Karlsen considered the saturation equation in the fractional flow formulation of the two-phase porous media flow equations, which is actually a convection dominated, degenerate convection–diffusion equation. Since then, various numerical techniques have appeared in the literatures of the inverse problem in porous media flow. One of the most utilized techniques for this problem is the ensemble Kalman Filter [6], and recently Hoel et al. [7] developed a multilevel version of this method. In [8], a contamination source identification problem in constant porous media flow is solved with a hierarchical Bayesian computation method. In [9], nuclear magnetic resonance imaging data are used in an inverse problem methodology to estimate the porous media flow functions. Berre et al. [10] solved the multi-level parameter structure identification problem for two-phase porous media flow problems using flexible representations, and Krüger et al. [11] estimated a spatially dependent diffusion function in porous media flow utilizing pressure and fluid rate data. Liu [12] and Nilssen et al. [13] respectively applied the wavelet multiscale-homotopy method and the augmented Lagrangian method to the reconstruction of a permeability field in the multiphase porous media flow.

The inverse problem discussed in this paper is to identify $q(\mathbf{x})$ in the following nonlinear convection-diffusion equation in the multiphase porous media flow:

$$u_t + \nabla \cdot (\varphi, \psi) - \nabla \cdot (q(\mathbf{x}) \cdot N(u) \nabla u) = s(\mathbf{x}, t), \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

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with the initial-boundary condition

$$\begin{cases} u(\mathbf{x}, 0) = \phi(\mathbf{x}), & \text{in } \Omega, \\ u(\mathbf{x}, t) = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{1.2}$$

where $u$ is the concentration field, $q$ is the permeability coefficient, $\mathbf{x}$ is the space variable, $t$ is the time variable, $N$ is the positive nonlinear function in the diffusion term, $s$ is the piecewise smooth source function, $\nabla \cdot (\varphi, \psi)$ is defined by $\nabla \cdot (\varphi, \psi) = \frac{\partial}{\partial x}\varphi(u) + \frac{\partial}{\partial y}\psi(u)$, and $\varphi$, $\psi$ are the nonlinear S-shaped flux functions of Buckley–Leverett type in the $x$, $y$ directions respectively.

Eq. (1.1) is related to the multiphase flow in porous media. The immiscible displacement of oil by water in a porous media with zero gravity effects can be described by a set of nonlinearly coupled partial differential equations [5]

$$\begin{cases} \nabla \cdot \upsilon = s_1(\mathbf{x}, t), \\ \upsilon = -q(\mathbf{x})\kappa(\mathbf{x}, u)(\nabla\mu - \theta(u)\nabla\zeta), \\ \eta(\mathbf{x})u_t + \nabla \cdot (\phi(u)\upsilon + \varphi(u)q(\mathbf{x})\nabla\zeta) - \nabla \cdot (q(\mathbf{x})N(u)\nabla u) = s_2(\mathbf{x}, t), \end{cases} \tag{1.3}$$

where $s_1$ and $s_2$ are the injection and production wells, $q$ is the permeability, $\upsilon$ is the total Darcy velocity, $\kappa$ is the total mobility of the phases, $\mu$ is the global pressure, $\theta$ is the density of the wetting phase, $\zeta$ is the height, $\eta$ is the porosity, $N$ is the nonlinear diffusion function, $\phi$ is the nonlinear S-shaped fractional flow function, $\varphi$ is defined by $\varphi(u) = (\theta(u) - \rho(u))\phi(u)\epsilon(\mathbf{x}, u)$, and $\rho$, $\epsilon$ are respectively the density and phase mobility of the nonwetting phase.

Eq. (1.1) is similar to Eq. (1.3) except for the convection term and the time derivative term. The time derivative terms in these two equations are equal if we assume that $\eta(\mathbf{x}) \equiv 1$. The difference in the convection terms is that Eq. (1.1) has no permeability dependence and no varying coefficient.

The inverse problem discussed in this paper can be viewed as a parametric data-fitting problem, and we can formalize such a problem in the framework of optimization where a functional defined in terms of discrepancy between observed and computed data is minimized over a model space. Generally, such problem is very challenging to solve, since:

(1) Ill-posedness: The solution does not depend continuously on the observed data. A minor disturbance of the observed data may cause large change on the solution of the inverse problem.
(2) Nonlinearity: Nonlinear dependence of the observed data with respect to the permeability field to be identified causes the presence of numerous local minima.
(3) Large computational cost: An inversion process needs tremendous forward computations, the computational cost is often very large.

Therefore, the key problem is how to quickly find a stable solution. To overcome the three main difficulties, we utilize a regularized nonlinear multigrid method.

Multigrid method is a specific form of multiresolution method that can be used to reduce the computational requirements and remove smooth error components of large numerical problems of partial differential equations [14–17]. This method works by recursively moving between different resolutions, thereby propagating information between coarse and fine scales. Multigrid method has been primarily used for solving forward problems, but more recently it has also been applied to a variety of inverse problems. Ascher and Haber [18] proposed an efficient multigrid method for the recovery of a coefficient function of an elliptic differential equation in three dimensions. Spiliopoulos et al. [19] applied a multigrid approach for the estimation of geometric anisotropy parameters from scattered spatial data that are obtained from environmental surveillance networks. Mewes et al. [20] studied the multigrid Monte Carlo solution of anisotropic seismic inversion. McCormick and Wade [21] applied multigrid methods to a linearized electrical impedance tomography problem, and Borcea [22] used a nonlinear multigrid approach to electrical impedance tomography based on a direct nonlinear formulation analogous to the full approximation scheme in nonlinear multigrid partial differential equation solvers. Ye et al. [23] formulated the multigrid approach directly in an optimization framework, and used the method to solve optical diffusion tomography problems. In related work, Oh et al. [24] formulated multigrid algorithms for the solution of a broad class of optimization problems arising from inverse problems. Importantly, both the approaches of Ye and Oh are based on the matching of objective functional derivatives at different grids.

In this paper, we discuss the use of the nonlinear multigrid method for the solution of inverse problem for the nonlinear convection-diffusion equation in the multiphase porous media flow. The forward problem is discretized by the finite-difference method at different grids, and the inverse problem is formulated as a sequence of optimization problems. Then, we dynamically adjust the objective functionals in these optimization problems, in order to make them consistent, and ultimately reduce the objective functional at the finest grid. The motivation for using the multigrid method is that it can provide several unique advantages. First, some form of iteration is usually imperative in nonlinear problems, so multigrid methods are well adapted for solving nonlinear problems due to their iterative nature [25]. Second, since the coarse grid problems involve reduced number of variables, the number of local minima is enormously reduced and those that remain are further apart from each other. The reason lies in that by multigrid decomposition, lots of local minima are degenerated to trivial points, and the risk of trapping in a local minimum is reduced, thus multigrid methods can improve the inversion result. Third, multigrid methods can greatly reduce computation because both the forward and inverse problems are more coarsely discretized at coarser grids.