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Some new spectral bounds for graph irregularity

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ABSTRACT

The irregularity of a simple graph $G = (V, E)$ is defined as

$$\mathrm{irr}(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|,$$

where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. This graph invariant, introduced by Albertson in 1997, is a measure of the defect of regularity of a graph. Recently, it also gains interest in Chemical Graph Theory, where it is named the third Zagreb index. In this paper, by means of the Laplacian eigenvalues and the normalized Laplacian eigenvalues of G, we establish some new spectral upper bounds for $\mathrm{irr}(G)$. We then compare these new bounds with a known bound by Goldberg, and it turns out that our bounds are better than the Goldberg bound in most cases. We also present two spectral lower bounds on $\mathrm{irr}(G)$.

1. Introduction

All graphs throughout this paper are finite, undirected and simple. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Denote by $d_G(v_i)$ (or simply, d_i) the degree of vertex v_i in G, $i = 1, 2, \ldots, n$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G. We also write $\Delta(G)$ and $\delta(G)$ for the maximum vertex degree and the minimum vertex degree of G, respectively. A graph G is said to be regular if $\Delta(G) = \delta(G)$.

The adjacency matrix of a graph G is $A(G) = (a_{ij})_{n\times n}$, where elements $a_{ij} = 1$ if the vertices v_i and v_j in G are adjacent, and $a_{ij} = 0$ otherwise. The Laplacian matrix of G is defined to be $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. If G has no isolated vertices, then the normalized version of $L(G)$ is defined as $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$. It is well known that both $L(G)$ and $\mathcal{L}(G)$ are positive semi-definite matrices, and hence their eigenvalues (always known as the Laplacian eigenvalues and the normalized Laplacian eigenvalues of G, respectively) are nonnegative real numbers, which could be ordered by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ and $\nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G)$, respectively. For more details regarding the Laplacian eigenvalues and the normalized Laplacian eigenvalues of a graph, one may refer to [5,6].

In 1997, Albertson [2] defined the imbalance of an edge $e = uv \in E(G)$ as $|d_G(u) - d_G(v)|$, and the irregularity of a graph G as

$$\mathrm{irr}(G) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|. \tag{1.1}$$

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Clearly, for a connected graph G, $\mathrm{irr}(G)=0$ if and only if G is regular, and for a non-regular graph G, $\mathrm{irr}(G)$ is a measure of the defect of regularity of G. In [2], Albertson first proved the following upper bound on $\mathrm{irr}(G)$:

$$\mathrm{irr}(G)\leq\frac{4n^3}{27},$$

which was later improved by Abdo et al. [1] as

$$\mathrm{irr}(G)\leq\left\lfloor\frac{n}{3}\right\rfloor\left\lceil\frac{2n}{3}\right\rceil\left(\left\lceil\frac{2n}{3}\right\rceil-1\right).$$

On the other hand, Zhou and Luo [17] showed that if G is a graph with n vertices and m edges, then

$$\mathrm{irr}(G)\leq\sqrt{m(nM_1(G)-4m^2)},\tag{1.2}$$

where $M_1(G)=\sum_{v\in V(G)}d_G(v)^2$, which is referred to as the first Zagreb index. Remark that this index and the second Zagreb index, defined by $M_2(G)=\sum_{uv\in E(G)}d_G(u)d_G(v)$, are two of the oldest and most investigated topological graph indices in Chemical Graph Theory, for details about the mathematical theory and the chemical applications of the Zagreb indices one can see [7,8,10,14–16] and the references cited therein. It is also worth mentioning that in [8] Fath–Tabar established several obvious connections between the sum in (1.1) and the (first and second) Zagreb indices, and based on these connections he named the sum in (1.1) the third Zagreb index and denoted it by $M_3(G)$. However, in the rest of the paper, we shall use its older name, i.e., the irregularity of a graph G, and denote it by $\mathrm{irr}(G)$.

There also have been some other upper bounds on $\mathrm{irr}(G)$ obtained by several authors such as Hansen and Mélot [11], Henning and Rautenbach [12], and Fath–Tabar [8]. Strictly speaking, these bounds, together with the Zhou–Luo bound (1.2), are noncomparable, but the Zhou-Luo bound (1.2) seems to be much sharper than the others for most graphs. Recently, by utilizing the spectral techniques, Goldberg made a further improvement upon the Zhou–Luo bound (1.2).

Theorem 1.1. *(Goldberg [9]) If G is a graph on n vertices and with m edges, then*

$$\mathrm{irr}(G)\leq\sqrt{m(nM_1(G)-4m^2)(\mu_1(G)/n)}.\tag{1.3}$$

In this paper, by means of the Laplacian eigenvalues and the normalized Laplacian eigenvalues of G, we establish some new spectral upper bounds for $\mathrm{irr}(G)$. We then compare these new bounds with the Goldberg bound (1.3), and it turns out that our bounds are better in most cases. We also give two spectral lower bounds on $\mathrm{irr}(G)$.

2. Preliminaries

Let K_n and $K_{a,b}$ $(a+b=n)$, as usual, denote the complete graph and the complete bipartite graph with n vertices, respectively. Denote by $G\cup H$ the vertex-disjoint union of two graphs G and H. In particular, kG stands for the vertex-disjoint union of k copies of G. Let $G\vee H$ be the graph obtained from $G\cup H$ by adding all possible edges joining the vertices in G with those in H. Denote by $\overline{G}$ the complement of a graph G.

We now present some results that will be used in the next section.

Lemma 2.1. *(see [13]) If G is a graph on $n\geq 2$ vertices, then $\mu_1(G)\leq n$ with equality if and only if $\overline{G}$ is disconnected.*

Lemma 2.2. *(see [5]) If G is a graph on $n\geq 2$ vertices, then $\nu_1(G)\leq 2$ with equality if and only if G has a non-trivial component which is bipartite.*

For a graph G, if its vertex set $V(G)$ could be partitioned into two non-empty subsets U and W such that each vertex in U has degree r and each vertex in W has degree s, then G will be called an (r, s)-semiregular graph. In particular, if $r=s$ in an (r, s)-semiregular graph, then G is r-regular.

Lemma 2.3. *(see [3]) If G is a graph on $n\geq 2$ vertices, then*

$$\mu_1(G)\leq\max_{uv\in E(G)}\left\{d_G(u)+d_G(v)\right\}.$$

Moreover, if G is connected, then the above equality holds if and only if G is a bipartite regular graph or a bipartite semiregular graph.

For any two disjoint subsets S, T of $V(G)$, denote by $e(S, T)$ the number of edges in G with one end in S and the other end in T. If $S\neq\emptyset$ and $T=V(G)\backslash S=\overline{S}$, then the following result provides upper and lower bounds on $e(S,\overline{S})$ involving the Laplacian eigenvalues of a graph G, see [6], p. 200, Theorem 7.5.1.

Lemma 2.4. *(see [6]) If G is a graph on $n\geq 2$ vertices and $\emptyset\neq S\subset V(G)$, then*

$$\mu_{n-1}(G)\frac{|S||\overline{S}|}{n}\leq e(S,\overline{S})\leq\mu_1(G)\frac{|S||\overline{S}|}{n}.\tag{2.1}$$

The maximum cut of a graph G is defined by

$$\mathrm{mc}(G)=\max\left\{e(S,\overline{S}):\emptyset\neq S\subset V(G)\right\}.$$