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A note on the bi-periodic Fibonacci and Lucas matrix sequences

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ABSTRACT

In this paper, we introduce the *bi-periodic Lucas matrix sequence* and present some fundamental properties of this generalized matrix sequence. Moreover, we investigate the important relationships between the bi-periodic Fibonacci and Lucas matrix sequences. We express that some behaviors of bi-periodic Lucas numbers also can be obtained by considering properties of this new matrix sequence. Finally, we say that the matrix sequences as Lucas, k-Lucas and Pell–Lucas are special cases of this generalized matrix sequence.

1. Introduction and preliminaries

There are so many studies in the literature that concern about the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin (see, for example [1,5,6,8,10,11,13,15], and the references cited therein). Especially, the Fibonacci and Lucas numbers have attracted the attention of mathematicians because of their intrinsic theory and applications.

Many authors have generalized Fibonacci and Lucas sequences in different ways. For example, in [1,5], the authors defined the bi-periodic Fibonacci sequence $\{q_n\}_{n\in\mathbb{N}}$ as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \tag{1.1}$$

and the bi-periodic Lucas sequence $\{l_n\}_{n\in\mathbb{N}}$ as in the form

$$l_n = \begin{cases} al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \\ bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \end{cases}, \tag{1.2}$$

where $q_0 = 0,\quad q_1 = 1,\ l_0 = 2,\ l_1 = a$ and a, b are nonzero real numbers. Also, in [1], Bilgici gave some relations between the bi-periodic Fibonacci and Lucas numbers as

$$l_n = q_{n-1} + q_{n+1}, \tag{1.3}$$

$$(ab+4)q_n = l_{n+1} + l_{n-1}. \tag{1.4}$$

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On the other hand, the matrix sequences have taken so much interest for different type of numbers ([2–4,7,9,12,14,16]). In [4], the authors defined the bi-periodic Fibonacci matrix sequence and obtained nth general term of this matrix sequence as

$$\mathcal{F}_n(a,b)=\begin{pmatrix}\left(\frac{b}{a}\right)^{\varepsilon(n)}q_{n+1} & \frac{b}{a}q_n\\ q_n & \left(\frac{b}{a}\right)^{\varepsilon(n)}q_{n-1}\end{pmatrix}, \tag{1.5}$$

where

$$\varepsilon(n)=\begin{cases}1, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}. \tag{1.6}$$

In addition, the authors found the Binet formula of the bi-periodic Fibonacci matrix sequence

$$\mathcal{F}_n(a,b)=A_1(\alpha^n-\beta^n)+B_1\left(\alpha^{2\lfloor\frac{n}{2}\rfloor+2}-\beta^{2\lfloor\frac{n}{2}\rfloor+2}\right), \tag{1.7}$$

where $A_1=\frac{[\mathcal{F}_1(a,b)-b\mathcal{F}_0(a,b)]^{\varepsilon(n)}[a\mathcal{F}_1(a,b)-\mathcal{F}_0(a,b)-ab\mathcal{F}_0(a,b)]^{1-\varepsilon(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)}$, $B_1=\frac{b^{\varepsilon(n)}\mathcal{F}_0(a,b)}{(ab)^{\lfloor\frac{n}{2}\rfloor+1}(\alpha-\beta)}$.

In the light of all these above material, the main goal of this paper is to investigate the relationships between the bi-periodic Fibonacci and bi-periodic Lucas matrix sequences. To do that, first, we define the bi-periodic Lucas matrix sequences, because it is worth to study a new matrix sequence related to less known numbers. Then, it will be given the generating function, Binet formula and summation formulas for this new matrix sequence. By using the results in Sections 2, we have a great opportunity to obtain new properties in Section 3.

2. The matrix sequence of bi-periodic Lucas numbers

In this section, we mainly focus on the matrix sequence of the bi-periodic Lucas numbers. In fact, we present the some properties and Binet formula of this matrix sequence. Also, we investigate various summations of this matrix sequence.

Now, we first define the bi-periodic Lucas matrix sequence as in the following.

Definition 2.1. For $n\in\mathbb{N}$ and $a,b\in\mathbb{R}-\{0\}$, the bi-periodic Lucas matrix sequence $\mathcal{L}_n(a,b)$ is defined by

$$\mathcal{L}_n(a,b)=\begin{cases}a\mathcal{L}_{n-1}(a,b)+\mathcal{L}_{n-2}(a,b), & n \text{ odd}\\ b\mathcal{L}_{n-1}(a,b)+\mathcal{L}_{n-2}(a,b), & n \text{ even}\end{cases} \tag{2.1}$$

with initial conditions $\mathcal{L}_0(a,b)=\begin{pmatrix}a & 2\\ 2\frac{a}{b} & -a\end{pmatrix}$, $\quad\mathcal{L}_1(a,b)=\begin{pmatrix}a^2+2\frac{a}{b} & a\\ \frac{a^2}{b} & 2\frac{a}{b}\end{pmatrix}$.

In the following theorem, we give the nth general term of the matrix sequence in (2.1) via the bi-periodic Lucas numbers.

Theorem 2.2. *For any integer $n\geq 0$, we have the matrix sequence*

$$\mathcal{L}_n(a,b)=\begin{pmatrix}\left(\frac{a}{b}\right)^{\varepsilon(n)}l_{n+1} & l_n\\ \frac{a}{b}l_n & \left(\frac{a}{b}\right)^{\varepsilon(n)}l_{n-1}\end{pmatrix}, \tag{2.2}$$

where $\varepsilon(n)$ is as in the Eq. (1.6).

Proof. The proof can be seen by using the induction method and Eq. (2.1). □

As a consequence of Theorem 2.2, we rewrite with a different approximation the Cassini identity which is given in [1].

Corollary 2.3. *The following equalities are valid for all positive integers:*

- *Let $\mathcal{L}_n(a,b)$ be as in (2.2). Then*

$$\det(\mathcal{L}_n(a,b))=(ab+4)\left(-\frac{a}{b}\right)^{1+\varepsilon(n)}. \tag{2.3}$$

- *Cassini identity can also be obtained using the bi-periodic Lucas matrix sequence. That is, by using Theorem 2.2 and the Eq. (2.3), we can write*

$$\left(\frac{b}{a}\right)^{\varepsilon(n+1)}l_{n+1}l_{n-1}-\left(\frac{b}{a}\right)^{\varepsilon(n)}l_n^2=(ab+4)(-1)^{n+1}.$$

Theorem 2.4. *For every $n\in\mathbb{N}$, the following statements are true:*

(i) $\mathcal{L}_{n-1}(a,b)+\mathcal{L}_{n+1}(a,b)=\frac{a}{b}(ab+4)\mathcal{F}_n(a,b)$,
(ii) $\mathcal{F}_{n-1}(a,b)+\mathcal{F}_{n+1}(a,b)=\frac{b}{a}\mathcal{L}_n(a,b)$.